Features

From Regular Pentagons to the Icosahedron and Dodecahedron via the Golden Ratio – II

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Introduction

In the previous article (At Right Angles, Issue 4, July 2019, pages 5-9) we saw how we could construct a regular pentagon using a ruler and compass, and discovered a nested sequence of pentagons that can be built up by extending the sides of a given regular pentagon (see Figure 1).

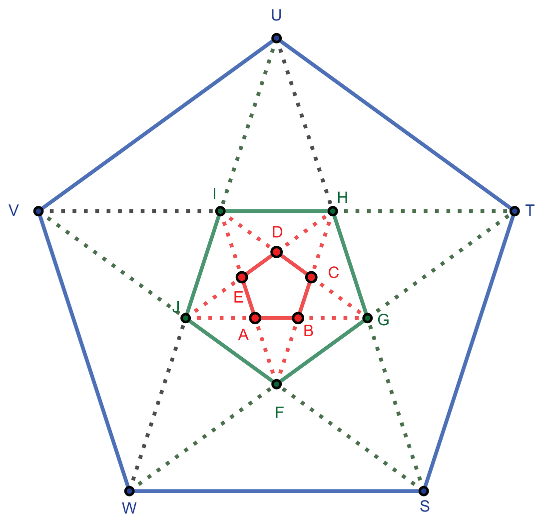

Figure 1.

We also calculated the edges and diagonals of each regular pentagon (see Figure 2)

Keywords: Regular pentagon, Golden Rectangle, Fibonacci sequence, icosahedron, dodecahedron, dual

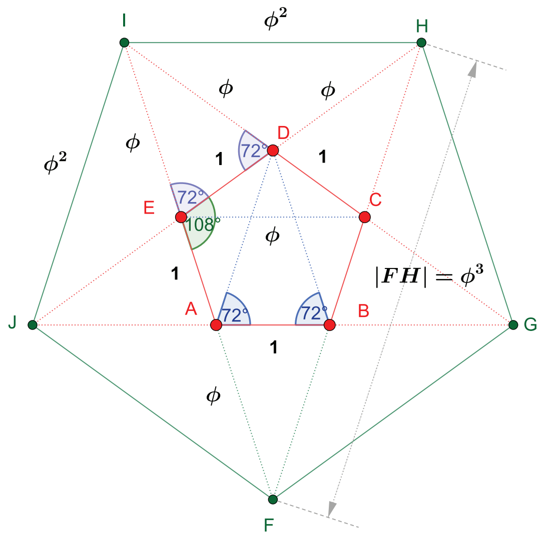

Figure 2.

and got the following infinite sequence of side, diagonal, side, diagonal,...

$$1,\ \phi,\ \phi^2,\ \phi^3, \ldots$$

where ϕ is the Golden Ratio.

In this article [1] we will see that this sequence leads to two famous sequences, the Fibonacci sequence and its cousin, the Lucas sequence. Moreover, we will see how the regular pentagon, the Golden Ratio, the icosahedron and the dodecahedron all come together!

The Fibonacci and Lucas sequences from a regular pentagon

Recall that the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$ and it satisfies the equation $x^2 - x - 1 = 0$, that is, $\phi^2 - \phi - 1 = 0$. We are going to use this equation to compute the powers of ϕ (we saw a hint of this in the last section of the previous article). Let us begin with ϕ^2. Of course we could compute $(\frac{1+\sqrt{5}}{2})^2$, but mathematicians need to be lazy if they can get away with it! We know $\phi^2 = \phi + 1$, so why not use that? This computation is much simpler! Therefore,

$$\phi^2 = 1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}.$$

What about ϕ^3? We saw in the previous article that $\phi^3 = \phi(\phi^2) = \phi(\phi + 1) = \phi^2 + \phi$. Hence,

$$\phi^3 = \frac{1+\sqrt{5}}{2} + \frac{3+\sqrt{5}}{2} = \frac{4+2\sqrt{5}}{2}.$$

We can similarly see that

$$\phi^4 = \phi^3 + \phi^2 = \frac{7+3\sqrt{5}}{2}.$$

The reader has perhaps begun to notice some patterns. From now on the number n shall refer to a whole number. First of all we can see by induction (keeping in mind that $\phi^0 = 1$) that

$$\phi^{n+1} = \phi^n + \phi^{n-1} \tag{1}$$

[1] I would like to thank Aashotosha Lele, Rishabh Suresh and Gautam Dayal for their suggestions and feed back while writing this article.

Let us denote $\phi^n = \frac{l_n + f_n\sqrt{5}}{2}$ with $n \geq 1$ and see if we can find a pattern in the computation of both l_n and f_n. From ϕ we get $l_1 = 1,\ f_1 = 1$ and from the fact that $\phi^2 = \phi + 1$, we get $l_2 = 3,\ f_2 = 1$. Now for $n > 1$, from Equation (1) we get:

$$\phi^{n+1} = \frac{l_{n+1} + f_{n+1}\sqrt{5}}{2} = \phi^n + \phi^{n-1} = \frac{l_n + f_n\sqrt{5}}{2} + \frac{l_{n-1} + f_{n-1}\sqrt{5}}{2} = \frac{(l_n + l_{n-1}) + (f_n + f_{n-1})\sqrt{5}}{2}, \tag{2}$$

giving us the formulas:

$$l_{n+1} = l_n + l_{n-1},\ n > 1 \tag{3}$$

$$f_{n+1} = f_n + f_{n-1},\ n > 1 \tag{4}$$

yielding $l_3 = 4,\ f_3 = 2,\ l_4 = 7,\ f_4 = 3$ and so on.

It is now time to pay attention to the other root of the equation $x^2 - x - 1 = 0$, namely $\psi = \frac{1-\sqrt{5}}{2}$. Notice ψ also satisfies the equation $\psi^2 = \psi + 1$. Using the same technique as above, we can see that

$$\psi^2 = 1 + \frac{1-\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$$

and

$$\psi^3 = \frac{4-2\sqrt{5}}{2}$$

and

$$\psi^4 = \frac{7-3\sqrt{5}}{2}$$

and so on. Just as in the case of ϕ^n, we get $\psi^n = \frac{l_n - f_n\sqrt{5}}{2},\ n \geq 1$ and

$$\psi^{n+1} = \psi^n + \psi^{n-1}, \text{where } \psi^0 = 1.$$

We can now create the following table:

$\phi = \frac{1+\sqrt{5}}{2}$	$\psi = \frac{1-\sqrt{5}}{2}$
$\phi^2 = \frac{3+\sqrt{5}}{2}$	$\psi^2 = \frac{3-\sqrt{5}}{2}$
$\phi^3 = \frac{4+2\sqrt{5}}{2}$	$\psi^3 = \frac{4-2\sqrt{5}}{2}$
$\phi^4 = \frac{7+3\sqrt{5}}{2}$	$\psi^4 = \frac{7-3\sqrt{5}}{2}$
$\phi^5 = \frac{11+5\sqrt{5}}{2}$	$\psi^5 = \frac{11-5\sqrt{5}}{2}$
$\vdots$	$\vdots$
$\phi^n = \frac{l_n+f_n\sqrt{5}}{2},\ n \geq 1$	$\psi^n = \frac{l_n-f_n\sqrt{5}}{2},\ n \geq 1$

The astute reader might have realised that adding and subtracting each row of the above table gives rise to some familiar sequences:

$\phi+\psi=l_1=1$	$\phi-\psi=f_1\sqrt{5}=\sqrt{5}$
$\phi^2+\psi^2=l_2=3$	$\phi^2-\psi^2=f_2\sqrt{5}=\sqrt{5}$
$\phi^3+\psi^3=l_3=4$	$\phi^3-\psi^3=f_3\sqrt{5}=2\sqrt{5}$
$\phi^4+\psi^4=l_4=7$	$\phi^4-\psi^4=f_4\sqrt{5}=3\sqrt{5}$
$\phi^5+\psi^5=l_5=11$	$\phi^5-\psi^5=f_5\sqrt{5}=5\sqrt{5}$
$\vdots$	$\vdots$
$\phi^n+\psi^n=l_n,\ n\geq 1$	$\phi^n-\psi^n=f_n\sqrt{5},\ n\geq 1$

So, the two sequences that are emerging are

$$l_1=1,\ l_2=3,\ l_3=4,\ l_4=7,\ l_5=11\ldots$$

and

$$f_1\sqrt{5}=\sqrt{5},\ f_2\sqrt{5}=\sqrt{5},\ f_3\sqrt{5}=2\sqrt{5},\ f_4\sqrt{5}=3\sqrt{5},\ f_5\sqrt{5}=5\sqrt{5}\ldots.$$

The first sequence

$$L=\{l_1,l_2,l_3,l_4,l_5,\ldots l_n,\ldots\}=\{1,\ 3,\ 4,\ 7,\ 11\ldots\}$$

is called the Lucas sequence. In the second sequence if we divide by $\sqrt{5}$ throughout, we get

$$F=\{f_1,f_2,f_3,f_4,f_5,\ldots f_n,\ldots\}=\{1,\ 1,\ 2,\ 3,\ 5\ldots\}$$

the famous Fibonacci sequence!

Both these sequences have the same generative principle: you start with two given terms, here $l_1=1, l_2=3$ and $f_1=f_2=1$ and then generate the sequences using the iterative Equations (2) and (3), namely $l_{n+1}=l_n+l_{n-1}$ and $f_{n+1}=f_n+f_{n-1},\ n>1$.

Let us now return to the n^{th} term of the Lucas sequence $l_n=\phi^n+\psi^n$ and the n^{th} term of the Fibonacci sequence $f_n=\dfrac{\phi^n-\psi^n}{\sqrt{5}}$. In other words

$$l_n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n \text{ and } \sqrt{5}f_n=\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now consider $\psi=\frac{1-\sqrt{5}}{2}$; it is easy to see that $-1<\psi<0$, and hence $\lim_{n\to\infty}\psi^n=0$. This then tells us that if n is large $l_n\approx\left(\frac{1+\sqrt{5}}{2}\right)^n$ and that $\sqrt{5}f_n\approx\left(\frac{1+\sqrt{5}}{2}\right)^n$, yielding the following amazing result

$$\frac{l_n}{l_{n-1}}\approx\frac{1+\sqrt{5}}{2}=\phi \text{ and } \frac{f_n}{f_{n-1}}\approx\frac{1+\sqrt{5}}{2}=\phi$$

the Golden Ratio! So for both the Fibonacci sequence and the Lucas sequence the ratio of successive terms approximates the Golden Ratio.

The Icosahedron and the Golden Ratio

It is time to now put together all that we have learned. A rectangle with side lengths 1 and ϕ is called a Golden Rectangle. Let us take three such Golden Rectangles, $ABCD$, $EFGH$, $IJKL$ and intersect them mutually perpendicular to each other in three dimensions along the x, y and z axes as shown in Figure 3.

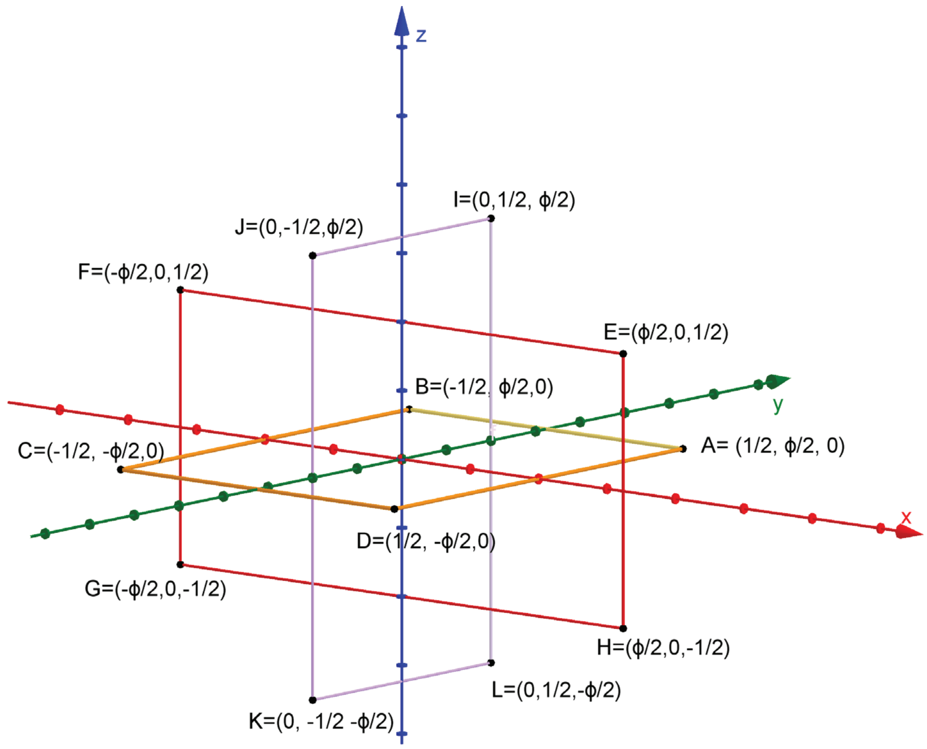

Figure 3.

Notice the special choice of coordinates for our 12 vertices:

$$\begin{array}{llll} A=(\frac{1}{2},\frac{\phi}{2},0), & B=(\frac{-1}{2},\frac{\phi}{2},0), & C=(\frac{-1}{2},\frac{\phi}{2},0), & D=(\frac{1}{2},\frac{-\phi}{2},0), \\ E=(\frac{\phi}{2},0,\frac{1}{2}), & F=(\frac{-\phi}{2},0,\frac{1}{2}), & G=(\frac{-\phi}{2},0,\frac{-1}{2}), & H=(\frac{\phi}{2},0,\frac{-1}{2}), \\ I=(0,\frac{1}{2},\frac{\phi}{2}), & J=(0,\frac{-1}{2},\frac{\phi}{2}), & K=(0,\frac{-1}{2},\frac{-\phi}{2}) \text{ and } & L=(0,\frac{1}{2},\frac{-\phi}{2}). \end{array}$$

This choice of coordinates ensures that we have Golden rectangles with the correct lengths. You might have also become aware of the amazing symmetry among the vertices.

We now claim that the vertices form the 12 vertices of a regular icosahedron! How do we construct the edges? This is quite intuitively obvious. Take the vertex E for example. There are 11 possible edges that can start at E, but we can reject EB, EC, EF, EG, EK and EL because they are longer than the unit edge EH. This leaves us the edges EA, ED, EH, EI and EJ. Don't worry we will show shortly that these edges do have length 1!

Recall that a regular icosahedron is one of the five Platonic solids. Platonic solids, also called the 'regular solids,' are 3-dimensional geometric solids whose faces are all congruent regular polygons (like equilateral triangles or squares) and in which the same number of polygons meet at each vertex. The amazing fact is that there are only 5 such Platonic solids. For a proof of this please see [3].

The regular icosahedron has 12 vertices, 30 edges and 20 faces (all of which are equilateral triangles). Moreover at each vertex 5 equilateral triangles meet. Notice that the polyhedron *ABCDEFGHIJKL* has 12 vertices, 30 edges and 20 faces and moreover, at each vertex 5 triangles meet. If we show that each of these triangles is equilateral (and hence congruent), then we would have established that *ABCDEFGHIJKL* is a regular icosahedron. We will do this by showing that all the 30 edges are of equal length.

Since six of the edge lengths come from our intersecting Golden Rectangles of breadth one, we need to show that all the other twenty four edges are of length one. But we don't need to do 24 calculations! Take

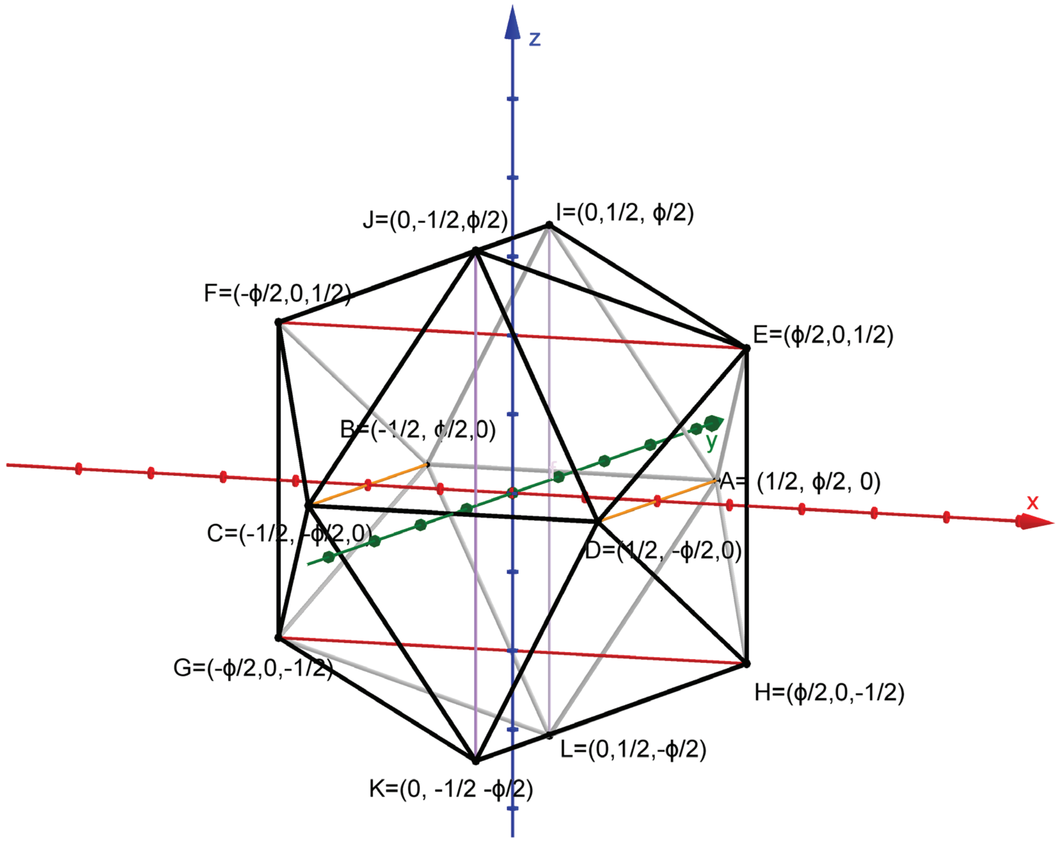

Figure 4.

any edge that forms the icosahedron other than the ones that come from the rectangle; so for example consider the edge *EA* as opposed to *EH*. (See Figure 4.) Using the Pythagorean formula in 3-D (the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + ((y_1 - y_2)^2 + (z_1 - z_2)^2)}$ we get:

$$|EA|^2 = (\frac{\phi - 1}{2})^2 + (\frac{-\phi}{2})^2 + (\frac{1}{2})^2 = \frac{2\phi^2 - 2\phi + 2}{4} = \frac{\phi^2 - \phi + 1}{2},$$

and from the equation $\phi^2 - \phi = 1$ we get:

$$(|EA|)^2 = 1.$$

You will notice that this very same computation works for every edge that is not a part of a Golden Rectangle, and so we are done!

What about the converse? That is, if we are given a regular icosahedron can we find three intersecting Golden Rectangles, whose vertices coincide with six of those of the regular icosahedron? Since regular icosahedra are essentially determined by their edge lengths, we know that any two regular icosahedra with the same edge length are congruent. Moreover, given an arbitrary regular icosahedron in 3-space, we can always move it (using translations and rotations) in such a way that it coincides with the regular icosahedron *ABCDEFGHIJKL*. The three Golden Rectangles then coincide with our original Golden Rectangles *ABCD*, *EFGH* and *IJKL*.

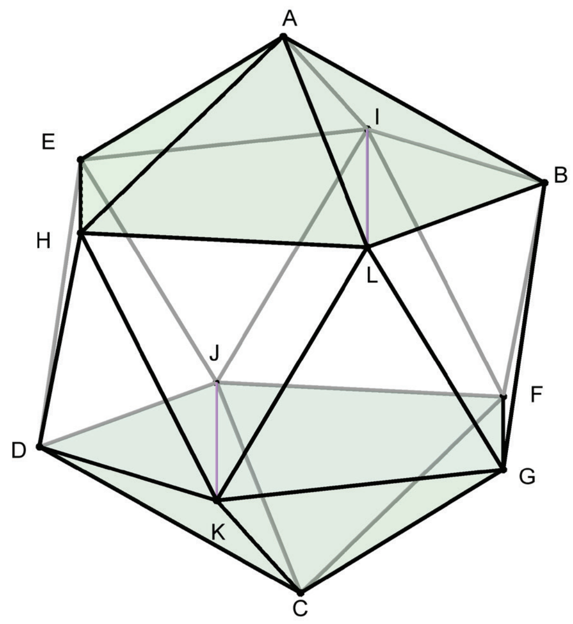

Figure 5.

We claimed that the regular pentagon is also part of the icosahedron. If you have not already noticed it, then you will see the regular pentagon show up in a regular icosahedron in quite a 'natural' manner. Every vertex of an icosahedron forms the apex of a pyramid whose base is a regular pentagon, a sort of 'pentagonal hat' as shown in Figure 5.

We have rotated the regular icosahedron in Figure 4 to obtain Figure 5, so that the vertex A is on top and it is the apex of the pyramid with base the regular unit pentagon $EHLBI$. Now IL is the diagonal of this pentagon and from part I of this article we know it has to be of length ϕ, which of course it is by construction!

The Dodecahedron and the Golden Ratio

Every Platonic solid comes with a dual (see [2] to learn more about duals). The icosahedron is the dual of the dodecahedron and vice-versa (for the rest of this article we will just say icosahedron and dodecahedron without the prefix 'regular', because all of the ones we refer to are going to be regular). One way to construct the dual is to take the centres of all the faces as vertices of the dual solid. So corresponding to the 12 vertices of the icosahedron are the 12 faces of the dodecahedron. The dodecahedron has 20 vertices (one for each face of the icosahedron) and 30 edges.

The following series of figures [2] shows how the regular dodecahedron is built up from the icosahedron. We begin with building four regular pentagons around the vertices A, E, I, and J. We have shaded the regular pentagon surrounding the vertex E.

[2]The reader might be curious as to how the coordinates of the twenty vertices of the regular dodecahedron are to be found, given Figure 4. It turns out that the simplest way is to first recognize that there is a sphere with centre the origin and radius $\frac{\sqrt{\phi+2}}{2} = \frac{\sqrt{5+\sqrt{5}}}{4}$ that envelops the icosahedron formed by the intersecting Golden Rectangles . That is, it passes through all twelve vertices $A, B, \ldots, L$. Let us call this sphere $\mathbb{S}$. Then the faces of the dodecahedron lie on the tangent planes to $\mathbb{S}$ passing through the 12 vertices $A, B, \ldots L$. To find the coordinates of the twenty vertices of the regular dodecahedron we need to first find the lines of intersection of suitable tangent planes and then find the points of intersection of suitable lines. Obviously this footnote is too small to fit in all the details and providing all the details is an article in its own right!

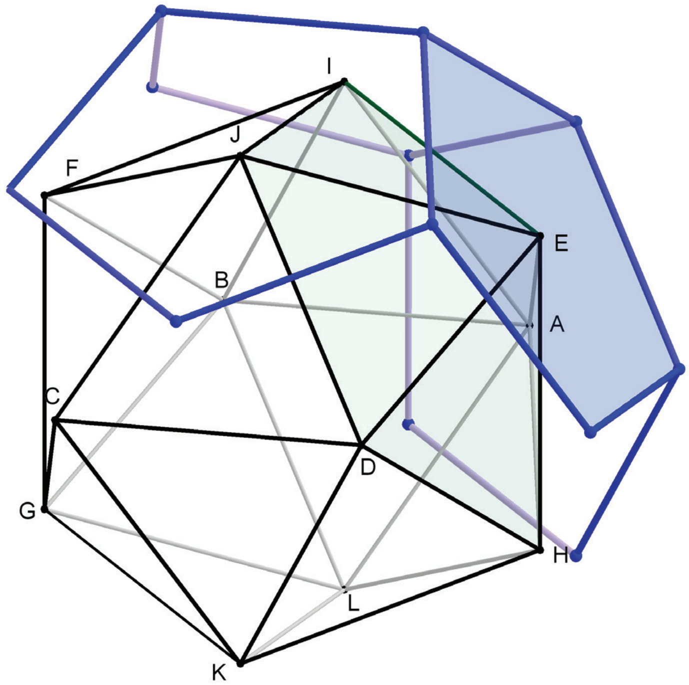

Figure 6.

We then add regular pentagons around the vertices H, D, K, and L.

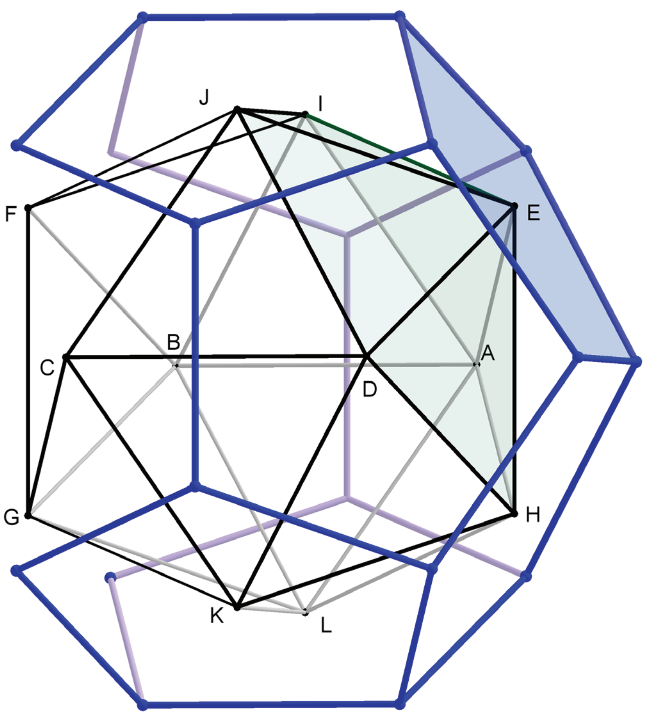

Figure 7.

And finally B, F, G, and C.

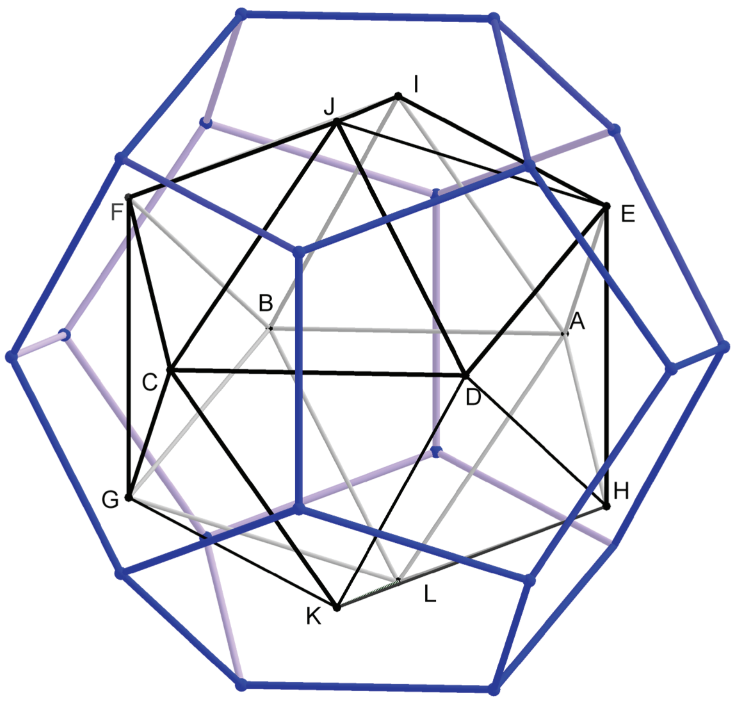

Figure 8.

We can now see how the Golden Rectangles, the icosahedron and the dodecahedron come together.

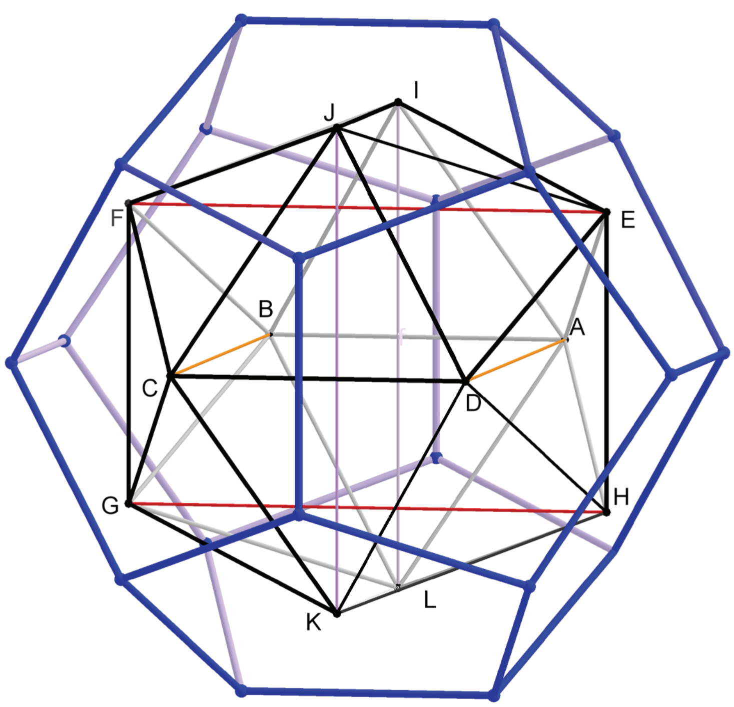

Figure 9.

We hope that in this two part series we have managed to convince the reader that the regular pentagon, the Golden Ratio, the Fibonacci sequence the icosahedron and the dodecahedron all come together so beautifully.

References

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