Brahmagupta's Theorem

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In this short note, we discuss a beautiful theorem first proved by the great seventh century Indian mathematician Brahmagupta—a theorem about a cyclic quadrilateral.

Theorem. *In a cyclic quadrilateral whose diagonals are perpendicular to each other, the line through the point of intersection of the diagonals which is perpendicular to one side bisects the opposite side.*

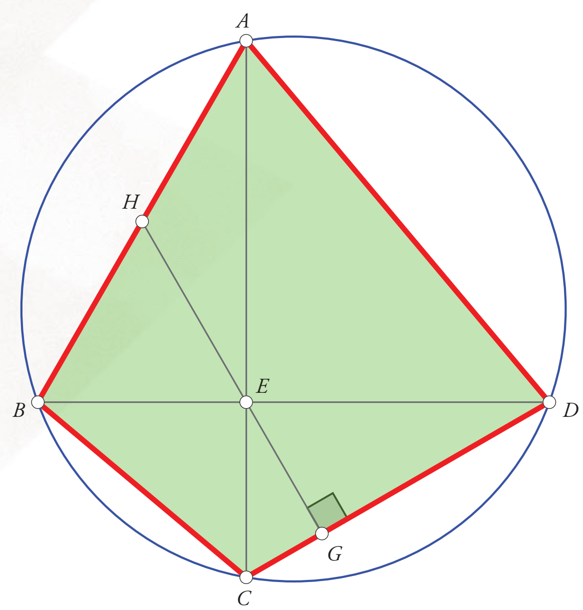

Figure 1.

Figure 1 illustrates what this states. $ABCD$ is a cyclic quadrilateral; E is the point of intersection of diagonals AC and BD. It is given that AC and BD are perpendicular to each other. Through E, a line EG is drawn, perpendicular to CD. It intersects AB at H. We must show that H is the midpoint of AB, i.e., $AH = HB$.

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Proof. In Figure 2, let x denote $\measuredangle CEG$, and let y denote $\measuredangle ECG$. Then x and y must be complements of one another (i.e., they add up to a right angle), since $\measuredangle EGC$ is a right angle. Observe that $\measuredangle AEH = x$ (vertically opposite angles), and $\measuredangle ABD = y$ (using the "angles in the same segment" theorem). Also, $\measuredangle BAE$ is the complement of $\measuredangle ABD$, since $\measuredangle AEB$ is a right angle. Since $\measuredangle ABE = y$, it follows that $\measuredangle BAE = x$. And since $\measuredangle BEH$ is the complement of $\measuredangle AEH$, it must be that $\measuredangle BEH = y$.

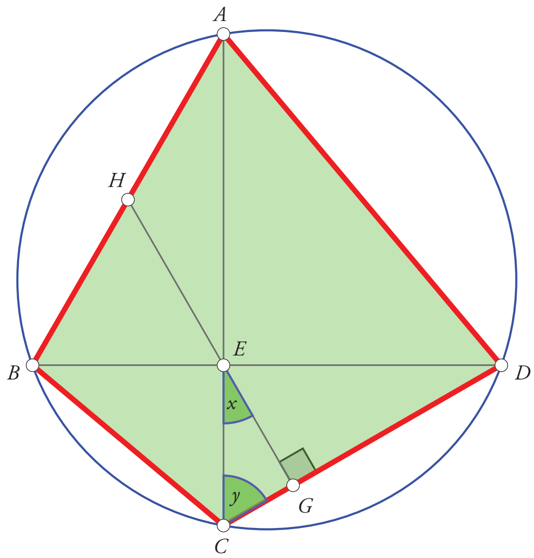

Figure 2.

Therefore we have $\measuredangle HEA = \measuredangle HAE$, and $\measuredangle HBE = \measuredangle HEB$. It follows that $HA = HE$, and $HE = HB$. Hence $HB = HA$, i.e., H is the midpoint of AB, as was to be proved. ■

Further remarks.

- The above proof is a nice illustration of the principle of transitivity: if $a = b$ and $b = c$, then $a = c$.
- Brahmagupta also showed that the area Δ of a cyclic quadrilateral with given sides a, b, c, d is given by the following beautifully symmetric formula:

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

  where $s = \frac{1}{2}(a+b+c+d)$ is the semi-perimeter of the quadrilateral.
- The above formula reduces to a familiar formula ('Heron's formula') for the area of a triangle in the case when $d = 0$, i.e., one side reduces to zero length (which means in effect that the quadrilateral has collapsed into a triangle).
- Brahmagupta was the first mathematician to state explicitly the arithmetic rules for operating with 0 and the rules for working with negative numbers.