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An Unusual Proof of the Centroid Theorem

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Introduction. In this article, I present an unusual proof of the Centroid Theorem. (The theorem states: *For any triangle, the three medians meet in a point. Moreover, the common point of intersection is a point of trisection of each median.*) The standard methods (see [3], pg 65 for a much shorter proof that uses the same base results as this one, or [1], pg 7 for one that uses Ceva's theorem) require nothing but elementary geometry. Another vector-based approach (see [2], pg 19) also exists. This one, however, makes use of an infinite geometric progression to achieve its result.

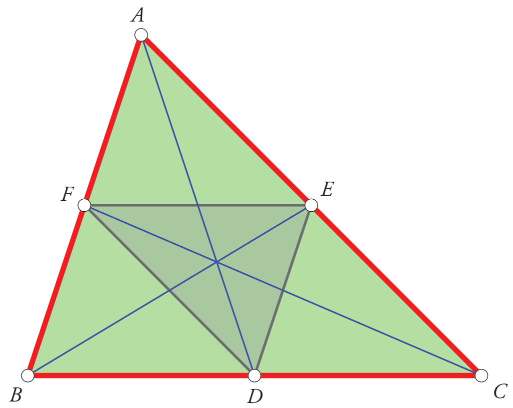
Background. The centroid is a point that has been known since antiquity [4]. It is the meeting point of the three medians of a triangle (lines from each vertex to the midpoint of the opposite side). Interestingly, this point is also the triangle's centre of mass.

It is not obvious that these lines should concur at all. Here I show that they do indeed intersect, and, moreover, that the point G of intersection is one that divides each median in the ratio 1 : 2.

Concurrence. Observe that the meeting point of each pair of medians of $\triangle ABC$ lies within $\triangle ABC$, and that triangles AFE, FBD, EDC and DEF are congruent to each other. (See Figure 1.)

Observe also that median AD bisects EF, BE bisects DF, and CF bisects DE (all these statements also require proof). So the medians of $\triangle DEF$ lie along the medians of $\triangle ABC$. Therefore, *the meeting point of each pair of medians of $\triangle ABC$ lies within $\triangle DEF$.*

Keywords: Median, centroid theorem, Ceva's theorem, vector, infinite geometric series



Given a triangle $\triangle ABC$, we find the midpoints D, E, and F of its sides and construct its medians and the medial triangle, $\triangle DEF$.

Figure 1. $\triangle ABC$ with its medians and the medial triangle, $\triangle DEF$

We can apply exactly the same logic to show that the meeting point of each pair of medians of $\triangle ABC$ lies within $\triangle D'E'F'$ (see Figure 2), the medial triangle of $\triangle DEF$. And then within $\triangle D''E''F''$. ...And so forth, until they are constrained to meet within a triangle of infinitesimal proportions (i.e., at a point), thus establishing concurrence. ■

Trisection. Now we show that the centroid divides each median in the ratio 1 : 2. We do this with the median FC, but the same logic would apply with the other medians.

Let F'' be the midpoint of $D'E'$. Then $FF'' = \frac{1}{4}FC$. Repeating the double-medial-triangle procedure that brought $\triangle D'E'F'$ into existence, we get $\triangle D''''E''''F''''$.

Then $F''F'''' = \frac{1}{4}FF'' = \frac{1}{16}FC$. Each successive iteration adds a smaller length one fourth the length of the previous one. Thus each time we add a new smaller medial triangle, we inch that much closer to G (unlabeled).

So we can write down a geometric series (with first term one fourth and an identical common ratio) that gives the distance FG exactly:

$$FG = FC\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right).$$

Recalling that the sum of the infinite geometric progression $a + ar + ar^2 + ar^3 + \cdots$ is $\frac{a}{1-r}$ if the common ratio r lies between -1 and 1, we deduce that the sum of the infinite progression within the brackets is $\frac{1/4}{1-1/4} = \frac{1}{3}$. This gives us the relation $FG = \frac{1}{3}FC$. ■

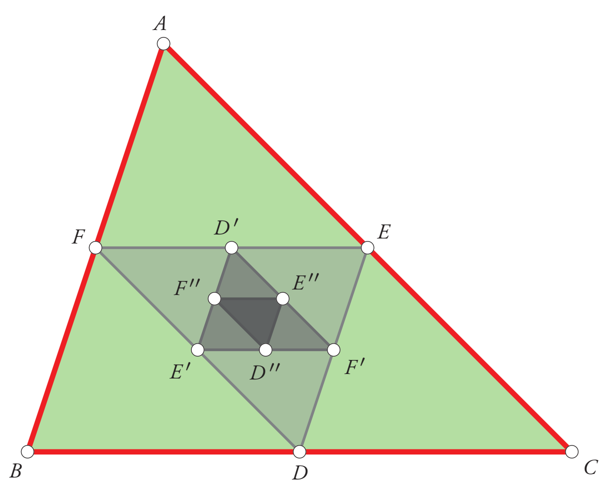
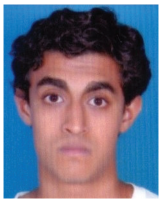


Figure 2. $\triangle D'E'F'$, $\triangle D''E''F''$, ...

References

1. Coxeter, H.S.M. and Greitzer, S.L., “Geometry Revisited.” Mathematical Association of America (1967).
2. Hirst, A., “Vectors in Two or Three Dimensions.” Modular Mathematics Series, Elsevier Science (1995).
3. Altshiller-Court, N., “College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle.” Dover Books on Mathematics, Dover Publications (2012).
4. Wikipedia, The Free Encyclopedia, “Centroid — Wikipedia.” https://en.wikipedia.org/w/index.php?title=Centroid&oldid=902707668.



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