A Modified PASCAL TRIANGLE

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The Pascal triangle is well known. Starting with a single 1 at the top, successive rows are obtained using a simple generating rule. Here is what we get as a result (we have shown only the first few rows):

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

The generating rule used is that *from the second row downwards, each term is the sum of the two numbers immediately above it, to its upper left and to its upper right.*

A Modified Generating Rule

Instead of this familiar rule, suppose we use the following rule: *from the second row downwards, each term is the sum of all the numbers located diagonally from it in the 'north-west' direction and in the 'north-east' direction.* (As earlier, we start with a single 1 at the top.) Here is what we get when we use this generating rule (we have shown only the first few rows):

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1

1 1

2 2 2

4 5 5 4

8 12 14 12 8

16 28 37 37 28 16

32 64 94 106 94 64 32

64 144 232 289 289 232 144 64

128 320 560 760 838 760 560 320 128

Figure 1

For example, we have:

- $5 = 2 + (2 + 1)$;
- $37 = (4 + 12) + (2 + 5 + 14)$;
- $94 = (8 + 28) + (2 + 5 + 14 + 37)$;
- $760 = (16 + 64 + 232) + (4 + 12 + 37 + 106 + 289)$;

and so on. The generating rule can be used to build the array indefinitely.

Observations and Conjectures

Looking closely at the resulting triangular array of numbers, we observe a large number of patterns. We list them here under the heading 'observations and conjectures.' While some of these are obviously true, the others have the status of 'conjectures awaiting proof.'

1. The array is symmetric about its central north-south axis, just as in the case of the Pascal triangle.
2. The elements of the left-most diagonal (and the right-most diagonal too, in view of the symmetry noted above) form the sequence 1, 1, 2, 4, 8, 16, 32, . . . and these are all powers of 2. If we leave out the initial 1, they form the successive powers of 2.
3. Labelling the topmost row as the zeroth row, the sum of the entries in the n-th row is equal to $2 \cdot 3^{n-1}$ (for $n \geq 1$).
4. Now let us arrange the array in a left-justified form so that it takes the shape of a right-angled triangle (see Figure 2; to bring out the property more clearly, we have drawn lines separating the rows and columns):

1								
1	1							
2	2	2						
4	5	5	4					
8	12	14	12	8				
16	28	37	37	28	16			
32	64	94	106	94	64	32		
64	144	232	289	289	232	144	64	
128	320	560	760	838	760	560	320	128

Figure 2

Now, using this array, let us compute the sums along the SW-NE (southwest-northeast) diagonals. (One such diagonal is indicated by the numbers shown in red font.) We obtain the following numbers:

$$1, 1, 3, 6, 15, 33, 78, 187, \ldots.$$

For example, we have $3 = 2 + 1$, $6 = 4 + 2$ and $15 = 8 + 5 + 2$. Let the n-th number in this sequence be denoted by a_n (so $a_1 = 1$, $a_2 = 1$, $a_3 = 3$, $a_4 = 6, \ldots$). We find the following relation connecting these numbers:

$$a_n = 3 \cdot (1+1 + 3 + \cdots + a_{n-2}) \text{ (for } n \geq 3),$$

i.e.,

$$a_n = 3 \sum_{i=1}^{n-2} a_i \qquad \text{(for } n \geq 3).$$

5. Consider the elements in the **second column** of the array in Figure 2 (note that this corresponds to the second NE-SW diagonal of the array in its original triangular form, Figure 1):

$$1, 2, 5, 12, 28, 64, 144, 320, \ldots.$$

Let the n-th number in the above sequence be denoted by b_n (so $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 12$, $b_5 = 28, \ldots$). We find the following relations connecting these numbers:

$$5 = (2 \times 2) + 1,$$
$$12 = (2 \times 5) + 2,$$
$$28 = (2 \times 12) + 4,$$
$$64 = (2 \times 28) + 8,$$
$$144 = (2 \times 64) + 16, \ldots$$

and, in general,

$$b_n = 2b_{n-1} + 2^{n-3} \qquad \text{for } n \geq 3.$$

6. Now consider the elements in the **third column** of the array in Figure 2 (note that this corresponds to the third NE-SW diagonal of the array in its original triangular form, Figure 1):

$$2, 5, 14, 37, 94, 232, 560, \ldots.$$

Let the n-th number in the above sequence be denoted by c_n (so $c_1 = 2$, $c_2 = 5$, $c_3 = 14$, $c_4 = 37$, $c_5 = 94, \ldots$). We find the following relations connecting these numbers:

$$14 = (2 \times 5) + 4,$$
$$37 = (2 \times 14) + 9,$$
$$94 = (2 \times 37) + 20,$$
$$232 = (2 \times 94) + 44,$$
$$560 = (2 \times 232) + 96, \ldots.$$

The pattern here is not readily seen but becomes visible when we focus on the sequence 4, 9, 20, 44, 96, ... We obtain:

$$9 = (2 \times 4) + 1,$$
$$20 = (2 \times 9) + 2,$$
$$44 = (2 \times 20) + 4,$$
$$96 = (2 \times 44) + 8, \ldots.$$

We again see the powers of 2. But to uncover this pattern, we had to go 'one level deeper.'

7. Consider the elements in the **fourth column** of the array in Figure 2 (note that this corresponds to the fourth NE-SW diagonal of the array in its original triangular form, Figure 1). To uncover the pattern here, we have to go still deeper. The elements here are:

$$4, 12, 37, 106, 289, 760, 1944, 4864, \ldots.$$

Let the n-th number in the above sequence be denoted by d_n (so $d_1 = 4$, $d_2 = 12$, $d_3 =37$, $d_4 =106$, $d_5 =289, \ldots$). We find the following relations connecting these numbers:

$$37 = (2 \times 12) + 13,$$
$$106 = (2 \times 37) + 32,$$
$$289 = (2 \times 106) + 77,$$
$$760 = (2 \times 289) + 182,$$
$$1944 = (2 \times 760) + 424,$$
$$4864 = (2 \times 1944) + 976, \ldots.$$

Next we have, focusing on the sequence 13, 32, 77, 182, 424, 976, . . .:

$$32 = (2 \times 13) + 6,$$
$$77 = (2 \times 32) + 13,$$

$$182 = (2 \times 77) + 28,$$

$$424 = (2 \times 182) + 60,$$

$$760 = (2 \times 424) + 128, \ldots.$$

Finally we have, focusing on the sequence 6, 13, 28, 60, 128, . . .:

$$13 = (2 \times 6) + 1,$$

$$28 = (2 \times 13) + 2,$$

$$60 = (2 \times 28) + 4,$$

$$128 = (2 \times 60) + 8, \ldots,$$

and we have reached the same pattern again, only it is one layer still further down.

8. Consider again the elements in the second column (Figure 2):

$$1, 2, 5, 12, 28, 64, 144, 320, \ldots.$$

Let us form the *partial sums* of this sequence, i.e., the numbers 1, 1+2, 1+2+5, ... We obtain the following numbers:

$$1, 3, 8, 20, 48, 112, 256, 576, \ldots.$$

Now form the partial sums of *this* sequence. We obtain:

$$1, 4, 12, 32, 80, 192, 448, 1024, \ldots.$$

Let the n-th number in the above sequence be denoted by e_n (so $e_1 = 1$, $e_2 = 4$, $e_3 = 12$, $e_4 = 32$, $e_5 = 80, \ldots$). The formula that generates this sequence appears to be $e_n = n \cdot 2^{n-1}$.

Remark. There appear to be deep connections between the powers of 2 and the modified Pascal triangle. Probably, there are many more such connections to be found.

Editorial comment. The reader will notice the central role of *observed patterns* in this article. In general, the ability to spot patterns—whether numerical or geometric—is vitally important in mathematics. This is so because conjectures are generally made on the basis of observed patterns, and we notice these patterns only through a close study of experimentally generated data. (This is the mathematical equivalent of experiments in science followed by analysis of data.) The reader will also notice that assertions made in the article have not been proved. Hence we refer to them as 'observations' or 'conjectures.' We welcome proofs from our readers and we shall attempt to provide proofs in subsequent issues of the magazine.

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