Low Floor High Ceiling Tasks

In-cyclic Quads

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Every triangle has a circumcircle and an incircle. But is that true for every quadrilateral? Definitely not; there are quads which are not cyclic (recall that a quad with a circumcircle is called 'cyclic'). But what about quads with incircles? In this Low Floor High Ceiling article, we not only explore the *if and only if* condition for a quad to have an incircle, but also reflect a certain duality. It is more fun if students are exposed to Tasks 1 and 2 before they learn the theorems related to cyclic quads.

It is a good idea to start each of these as compass-straight edge construction challenges. Once students get an idea on how to construct, it is a very good idea to replicate them using GeoGebra. That will eliminate the possibilities or doubts whether a point is actually on the circle or just appears to be so. It is not important for students to be exposed to the term cyclic quads for this exploration.

The first task is an easy one of constructing circles around different quads.

Task 1: Circumcircles

This initial task is all about finding circumcircles around various quads.

a. Construct any square ABCD. Find the intersection O of its diagonals. Construct a circle with centre O and radius OA. What do you observe?

b. Repeat the same for any rectangle.

c. Repeat for any rhombus.

d. Repeat for any parallelogram.

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e. Repeat for any isosceles trapezium ABCD with AB || CD.

f. Construct the perpendicular bisectors of AB and AD of the above isosceles trapezium and let them intersect at X. Construct a circle with centre X and radius XA. What do you observe?

Teacher Note: This is to understand that rectangles and isosceles trapeziums are cyclic whereas general rhombi and general parallelograms may not be. It may be worth exploring when a rhombus can be cyclic. This can be explored with GeoGebra by fixing the sides and varying the angles. Students may be asked to extrapolate their findings and form a conjecture when a parallelogram can be cyclic. Then they can verify their claim with the help of GeoGebra.

In the case of an isosceles trapezium, it may be difficult initially since the centre is not specified. The last part provides the centre and the radius. With their help, students should be able to conclude that an isosceles trapezium also has a circumcircle. The centre need not always be inside the trapezium.

The second task is the reverse one where a circle is the starting point. This demands more construction skills and therefore can be an assessment of the students' understanding of properties of the familiar quads.

Task 2: Quads inside a circle: Draw any circle to begin.

a. Construct a square in the circle such that the vertices are on the circle. Is it unique? What is the ratio of the radius of the circle to the side of the square?

b. Construct a rectangle in the circle in the same manner. Is it unique? If not, construct another with a different aspect ratio. How is the radius related to the sides in each case?

c. Can you construct a trapezium inside the circle? What do you observe?

d. Can you construct a kite inside the circle? How much are the equal angles?

Teacher Note: Given any circle, squares of only one size can be inscribed. The radius: side of the square would be $1 : \sqrt{2}$. So given any circle, the squares that can be inscribed are congruent to each other. However multiple aspect ratios, i.e., length : width are possible for rectangles inscribed within a given circle. If a and b are the sides of an inscribed rectangle and r is the radius of the circle, then $a^2 + b^2 = 4r^2$. For each positive $a < 2r$, there is a unique positive b. But there are infinitely many such pairs (a, b) and each corresponds to a different rectangle. No two such rectangles are congruent. As expected, the inscribed trapezium would be isosceles. It is a good idea to let students reason it out. Note the appearance of kite in this task and the kind of equal angles it has. [This is also the kite that maximizes area among all kites with the same sides. Why?]

We delve into the incircles with the next two tasks which are similar to the previous ones.

Task 3: Incircles

Now we get into the incircles.

a. Construct a square ABCD and the intersection point O of its diagonals. Drop a perpendicular OE from O on AB. Construct a circle with centre O and radius OE. What do you observe? What is the ratio OE : AB?

b. Repeat the same for any rhombus

c. Repeat for any rectangle

d. Repeat for any parallelogram

e. Repeat the same for any kite

f. If $\angle A$ and $\angle C$ are the equal angles of a kite ABCD, construct the bisector of $\angle A$ and let it intersect BD at X. drop a perpendicular XE from X on AB. Construct a circle with centre X and radius XE. What do you observe?

Teacher Note: As in Task 1, we start with the quads and try to construct incircles. The square obviously has one, such that the inradius : side of the square = 1 : 2. Rhombi have incircles whereas rectangles and parallelograms do not. For a kite, the point of intersection of the diagonals is not the incentre. So, the last part indicates the incentre. It is a good idea to discuss why the point of intersection of the diagonals did not work for the kite. Certain parallels can be drawn based on the incentre of a triangle.

The next task, like Task 2, starts with the circle and poses higher levels of construction challenges.

Task 4: Quads outside a circle: Draw a circle to begin.

a. Construct a square around it so that each side of the square touches the circle. Is the square unique? What is the ratio of the radius of the circle to the side of the square?

b. Repeat the same for any rhombus. Is it unique? If not, construct another with different angles.

c. Can you construct a kite?

d. Can you construct an isosceles trapezium ABCD such that AB || CD and AD = BC? Extend AB to A' such that BA' = CD and AD to E such that DE = BC. What kind of triangle is ΔAA'E?

Teacher Note: This is a reverse activity (like Task 2) to understand that certain quads can be wrapped around circles. The constructions will demand some level of critical thinking and problem solving. The circumscribed square is unique for a given circle since radius : side = 1 : 2. However the same circle can have several (actually infinitely many) rhombi around it. Any arbitrary point outside the circle can be the vertex of a circumscribing rhombus. The construction can be fun and challenging. If students are familiar with drawing tangent to a circle, this will provide a good practice of that skill. Otherwise, this may be a good point to introduce that theorem. For a kite, there are more degrees of freedom. Any two points outside the circle and collinear with the centre can generate a circumscribed kite.

Note the reappearance of isosceles trapezium. The construction demands visualization. There are infinitely many isosceles trapeziums circumscribing any given circle. Any line segment shorter than the diameter and touching the circle at its midpoint would generate an isosceles trapezium. The triangle should be isosceles. Check the next task to know why!

Task 5: Draw any circle. Construct any quad ABCD around it such that all four sides touch the circle. Prove that AB + CD = AD + BC

Teacher Note: This follows easily from tangent properties and observing that each side is the sum of two tangents.

Task 6: Construct two equal lines XY and PQ. Find any two points Z and R on XY and PQ respectively such that XZ < ZY and PR < RQ.

a. Construct a convex quad ABCD such that AB = XZ, AD = PR, CD = ZY and BC = RQ. Note that you can choose any angle ∠A < 180°. Compare AB + CD and AD + BC.

b. Construct the angle bisectors of any two consecutive angles. Let the bisectors intersect at O. Drop a perpendicular OE from O to AB. Construct a circle with centre O and radius OE. What do you observe?

c. Is this the converse of Task 5? Make a conjecture.

d. Prove your conjecture.

Note on the construction: If one tries to draw a quad ABCD such that AB + CD = AD + BC with four arbitrary side-lengths satisfying the given condition and any given angle, then it is possible that the quad may not close (if ∠A is too large and/or if BC and CD are too small). Also, one must get a convex quad and therefore must choose C to be the point of intersection of the arcs (centred at B and D) further from A. Now any quad with the given side-sum condition, must have the shorter two sides consecutive. So, if one starts with these shorter sides as AB and AD, then the quad would definitely close for any choice of $\angle A < 180°$. Therefore, the conditions $XZ < ZY$ and $PR < RQ$ have been included.

Teacher Note: This is a way to establish the reverse i.e. if $AB + CD = AD + BC$ in a convex quad ABCD, then it has an incircle. The proof can be done in the following way:

- Let the angle-bisectors of $\angle A$ and $\angle B$ meet at I, drop perpendiculars from I to all sides – $IE \perp AD$, $IF \perp AB$, $IG \perp BC$ and $IH \perp CD$, and show that $IE = IF = IG$
- We need to show $IH = IE$
- Suppose $IH \geq IE$ (1) $\Rightarrow DH \leq DE$ (2) using Pythagoras
- $DE = AD - AE = (AB + DC - BC) - AE$
  $= (AF + BF) + DC - (BG + GC) - AE$
  $= DC - GC$ i.e. $DE = DC - GC$ (3)
- $DH = DC - HC$ (4)
- (2) – (4) $\Rightarrow HC \geq GC \Rightarrow IH \leq IG = IE$ (5)
- (1) and (5) $\Rightarrow IH = IE$ i.e. I is equidistant from all four sides, allowing an incircle to be drawn.

Supposing $IH \leq IE$ would lead to the same conclusion with all the inequalities reversed.

Closing reflections: Recall that the condition for a quad to be cyclic is that opposite angles are supplementary. That is equivalent to saying that the sums of opposite angle-pairs are equal, i.e., $\angle A + \angle C = \angle B + \angle D$ (= 180°). This is very similar to the condition for incircle where angles have been replaced by sides! It is also worth noting that the perpendicular bisectors of sides meet at the circumcentre while the angle-bisectors do at the incentre – another exchange of sides and angles – the duality!!

While squares, rectangles, rhombi and parallelograms get their fair share in the syllabus, kites and isosceles trapeziums do not. Here are some of the duality that we observed:

	Kites	Isosceles trapeziums (IT)
1.	Two pairs of equal adjacent sides	Two pairs of equal adjacent angles
2.	Line of symmetry passing through vertices	Line of symmetry passing through sides
3.	Have incircle	Have circumcircle
4.	Some kites have circumcircles	Some ITs have incircles
5.	Rhombi = equilateral quads are special kites	Rectangles = equiangular quads are special ITs
6.	Kites ∩ parallelograms = rhombi	ITs ∩ parallelograms = rectangles
7.	Rhombi with circumcircles = squares	Rectangles with incircles = squares

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