Problem Corner

Middle School Problems on AVERAGES

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Problems

Problem VIII–3–M.1

A trader sells two articles at the same price. On the first he gains 'p' percent ($0 < p < 100$) over his cost price, while he suffers a loss of 'p' percent on the second article. On the whole does he gain or lose? Express his overall gain/loss percentage in terms of 'p.' You could also consider the situation as one of fractional gain/loss with $\frac{p}{100} = x$, say.

Problem VIII–3–M.2

A businessman drives from city A to city B and returns to city A again by the same route every day, driving at a uniform speed both ways. One day he finds that he has to reduce his speed on the onward journey by 'p' percent. However, on the return journey he is able to drive at a speed 'p' percent higher than his usual speed. On this day, does he take more/less time than usual for the round trip? Express the increase/decrease percentage in time taken in terms of 'p.' As earlier, you could think of fractional increase/decrease with $\frac{p}{100} = x$, say ($x < 1$).

Problem VIII–3–M.3

A shop sells chocolate bars of two types priced at ₹ P and ₹ Q per piece. I spend equal *sums of money* buying the two types of bars. What is the average cost of a bar?

Problem VIII–3–M.4

Two cubes of equal *mass* but made of two different metals of densities x and y are fused together. What is the overall density of the object thus formed?

Problem VIII–3–M.5

A person makes a journey from city A to city B in three parts. He travels the first third of the way at a uniform (or average) speed of a km/hr. He travels the second third of the way at a speed of b km/hr and the remainder of the journey at c km/hr. What is the average speed for the entire trip?

Problem VIII–3–M.6

The population of rabbits in a woodland area increased by 20% during a certain year, decreased by 5% the next year and increased by 10% the following year. What is the average growth rate over the three years? In other words, what is the percentage change applied uniformly over the three years that leads to the same end result?

Solutions

Problem VIII-3-M.1

It would be easier to work with fractions rather than percentages. Let S denote the selling price of either article. Then the cost price of the first article is $\frac{S}{1+x}$, while the cost price of the second article is $\frac{S}{1-x}$. The total cost price is then

$$\frac{2S}{1-x^2},$$

which is greater than $2S$, the total selling price, indicating an overall loss. The fractional loss, i.e., loss amount divided by total cost price is then x^2 . As a percentage it is $\left(\frac{p}{100}\right)^2$ percent.

Problem VIII-3-M.2

We again work with fractions. Let us take the distance between the cities to be d and the person's usual driving speed to be s. The time taken for a normal round trip is then $\frac{2d}{s}$. The time taken on the day when the speeds for onward and return journeys are different is

$$\frac{d}{s(1-x)}+\frac{d}{s(1+x)}=\frac{2d}{s(1-x^2)},$$

which is greater than $\frac{2d}{s}$, the usual trip time. The fractional increase in time works out to be $\frac{x^2}{1-x^2}$.

Note: In both the above situations, the quantity $\left(1-x^2\right)$ plays a key role. This quantity is always less than 1. This fact crops up in other similar situations. For instance, if you increase the dimension of a square in one direction by a certain fraction/percentage and decrease the perpendicular dimension by the same fraction/percentage the area always decreases.

Problem VIII-3-M.3

Let the sum of money spent on each type of chocolate bar be S. Then the number of bars purchased of type 1 is $\frac{S}{P}$ while the number of bars of type 2 is $\frac{S}{Q}$. Therefore the average price of a bar is

$$\frac{\text{total money spent}}{\text{total number of bars}}=\frac{2S}{\frac{S}{P}+\frac{S}{Q}}=\frac{2PQ}{P+Q}.$$

Problem VIII-3-M.4

This problem is similar to the above. Let the mass of each part be M. Then the volumes of the two parts are $\frac{M}{x}$ and $\frac{M}{y}$. The average density is

$$\frac{\text{total mass}}{\text{total volume}}=\frac{2M}{\frac{M}{x}+\frac{M}{y}}=\frac{2xy}{x+y}.$$

Note: In the above two problems, what we find is known as the *harmonic mean*. If the *number of bars* of the two types were the same in Problem 3 or the *volumes* of the two parts were the same in Problem 4, we would just take the arithmetic mean.

Problem VIII-3-M.5

Average speed for the journey $= \dfrac{\text{total distance}}{\text{total time}}$. Taking the distance from city A to city B to be $3s$, this equals

$$\frac{3s}{\frac{s}{a}+\frac{s}{b}+\frac{s}{c}}=\frac{3abc}{ab+bc+ca}.$$

This is the harmonic mean of a, b, c. Note that if $a=b=c$, then the expression simplifies to a.

Now solve a similar problem where the journey is divided into four parts. Do you see a pattern in the answers?

Problem VIII-3-M.6

If P_O and P_F are the initial and final populations, we have

$$P_F=P_O\left(1+\frac{20}{100}\right)\left(1-\frac{5}{100}\right)\left(1+\frac{10}{100}\right).$$

Then if a is the required average growth percentage, we should have

$$P_F=P_O\left(1+\frac{a}{100}\right)^3.$$

That is, $\left(1+\frac{a}{100}\right)^3=1.2\times 0.95\times 1.1$, or $1+\frac{a}{100}=\sqrt[3]{1.2\times 0.95\times 1.1}$.

Thus, $1+\frac{a}{100}=1.078$ approximately (1.078 is the geometric mean of 1.2, 0.95 and 1.1), giving $a = 7.8\%$ approximately.