Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VIII-3-S.1

From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Determine the length of a side of the octagon.

Problem VIII-3-S.2

Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1, respectively, and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is 40°, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees? (See Figure 1.)

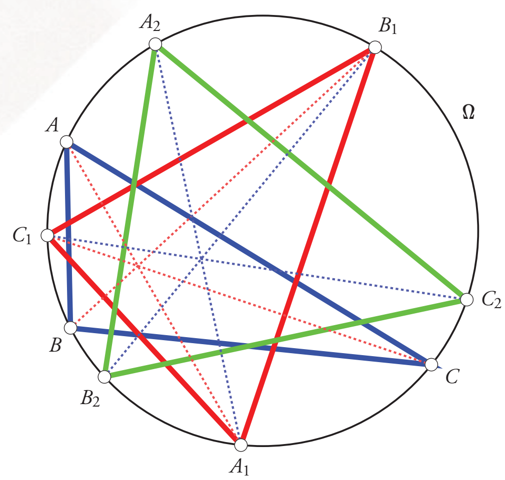

Figure 1.

Problem VIII-3-S.3

The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of ABC. Determine the angles of the triangle ABC.

Keywords: Octagon, circumcircle, internal bisector, least common multiple (LCM), prism

Solutions of Problems in Issue-VIII-2 (July 2019)

Solution to problem VIII-2-S.1

Let $ABCD$ be a parallelogram. Suppose K is a point such that $AK = BD$ and let M be the midpoint of CK. Prove that $\measuredangle BMD = 90^\circ$. [*Tournament of Towns*] (See Figure 2.)

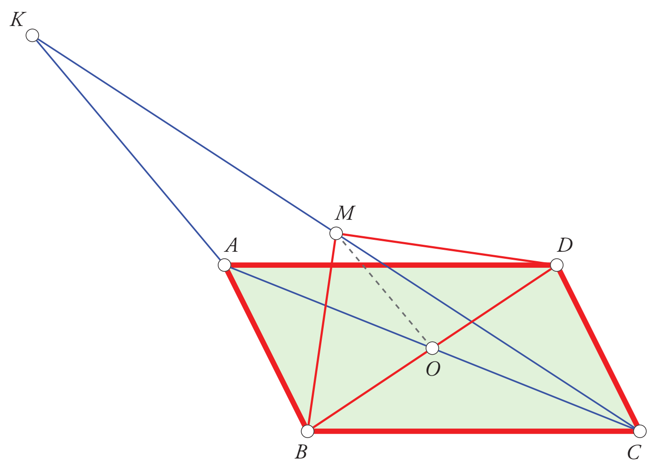

Figure 2.

Let $O = AC \cap BD$. Then in triangle ACK (possibly degenerate), M is the midpoint of CK and O is the midpoint of AC. Thus $OM = \frac{1}{2}AK = \frac{1}{2}BD$ and hence in triangle BMD, $BO = OM = OD$, implying $\measuredangle BMD = 90^\circ$.

Solution to problem VIII-2-S.2

Let A be a finite non-empty set of consecutive positive integers with at least two elements. Is it possible to partition A into two disjoint non-empty sets X and Y such that the sum of the least common multiples of the numbers in X and Y is a power of 2?

The answer is NO.

Any number in A can be written (uniquely) as $2^k m$ for some non-negative integer k and some odd positive integer m. Since there are finitely many elements in A, there are only finitely many non-negative integers k such that 2^k divides some element of A. Let l be the **largest** positive integer such that 2^l divides some element z of A.

As the elements of A are consecutive positive integers, l and z are unique. Without loss of generality we may assume that $z \in X$. Then the LCM of the elements of X is divisible by 2^l but the LCM of the elements of Y is not. Therefore their sum cannot be a power of 2.

Solution to problem VIII-2-S.3

The vertices of a prism are coloured using two colours, so that each lateral edge has its vertices differently coloured. Consider all the segments that join vertices of the prism and are not lateral edges. Prove that the number of such segments with endpoints differently coloured is equal to the number of such segments with endpoints of the same colour. [*Romanian Math Competition*]

Let n be the number of sides of the prism's base. Then the number of segments to be considered is

$$\binom{2n}{2} - n = \frac{2n(2n-1)}{2} - n = 2\left(n^2 - n\right).$$

Let a be the number of the vertices of the upper face which have the first colour and $b = n - a$ the number of the vertices of the upper face which have the second colour. Then the lower face has b points of the first colour and a points of the second colour.

The number of segments with endpoints differently coloured one on the upper face and one on the lower face is $a^2 + b^2 - n$. The number of segments with endpoints differently coloured and on the same face is $2ab$. So, the total number of segments with endpoints differently coloured is $(a+b)^2 - n = n^2 - n$, which is exactly half of the number of all the segments.

(A similar solution was sent to us by Rakshitha of Mangaluru.)

Solution to problem VIII-2-S.4
Let $a, b, c, d \in [0, 1]$. Prove that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+d} + \frac{d}{1+a} + abcd \leq 3. \qquad [\textit{Romanian Math Competition}]$$

Since $0 \leq a, b, c, d \leq 1$, it follows that $a, b, c, d \geq abcd$, and hence that

$$\begin{aligned}
&\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+d} + \frac{d}{1+a} + abcd \\
&\leq \frac{a}{1+abcd} + \frac{b}{1+abcd} + \frac{c}{1+abcd} + \frac{d}{1+abcd} + abcd \\
&= \frac{a+b+c+d}{1+abcd} + abcd.
\end{aligned}$$

Using repeatedly the inequality

$$x + y \leq 1 + xy \quad \forall x, y \in [0, 1]$$

(this is equivalent to $(1-x)(1-y) \geq 0$ and is therefore true), we get

$$\begin{aligned}
a + b + c + d &\leq 1 + ab + 1 + cd = ab + cd + 2 \\
&\leq 1 + abcd + 2 = abcd + 3.
\end{aligned}$$

Hence

$$\frac{a+b+c+d}{1+abcd} + abcd \leq \frac{3+abcd}{1+abcd} + abcd,$$

i.e.,

$$\frac{a+b+c+d}{1+abcd} + abcd \leq 1 + \frac{2}{1+abcd} + abcd.$$

Let $x = abcd$; then $0 \leq x \leq 1$, so it is enough to prove the following inequality for all $x \in [0, 1]$:

$$1 + \frac{2}{1+x} + x \leq 3,$$

that is,

$$\frac{2}{1+x} \leq 2 - x, \quad \text{i.e.,} \quad 2 \leq 2 + x - x^2,$$

or $x(1-x) \geq 0$, which is clearly true. Hence the stated inequality follows.

Solution to problem VIII-2-S.5

Let a and n be positive integers such that

$$\mathit{Frac}\left(\sqrt{n+\sqrt{n}}\right) = \mathit{Frac}\left(\sqrt{a}\right).$$

Prove that $4a+1$ is a perfect square. (Here Frac (x) = the fractional part of x.) [*Romanian Math Competition*]

The given condition is equivalent to $\sqrt{n+\sqrt{n}} = \sqrt{a} + k$, where k is an integer. This leads to

$$n + \sqrt{n} = a + 2k\sqrt{a} + k^2,$$
$$\therefore \quad \sqrt{n} = 2k\sqrt{a} + b, \quad \text{where } b = k^2 - n + a;$$

note that b is an integer. We now obtain

$$n = 4k^2a + b^2 + 4kb\sqrt{a},$$

which implies that $kb\sqrt{a}$ is a rational number.

If $\sqrt{a}$ is rational, then a is a perfect square, as is $n + \sqrt{n}$. This implies that $n = m^2$ for some natural number m. The hypothesis thus implies that $m^2 + m$ is a perfect square. But from the obvious inequality $m^2 \leq m^2 + m < (m+1)^2$, we obtain $m = 0$, that is $n = 0$, which is a contradiction (as we had supposed that n is a positive integer).

So, we must have $kb = 0$. If $b = 0$, we get $n = k^2 + a$ and also $n = 4k^2a$, hence

$$a = \frac{k^2}{4k^2 - 1} < 1,$$

which is again a contradiction (as we had supposed that a is a positive integer). The only possibility is $k = 0$, and thus $n = b^2$ and $a = b + n$, hence $a = b^2 + b$ and $4a + 1 = (2b+1)^2$, which is a perfect square, as claimed.