Features

# Tiresome Paths, Water Gates & Euler's Formula

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A hallmark of mathematics is its power to look at seemingly different problems with the same eyes and find a common idea which resolves both. It is not surprising that the two problems we discuss here, about routes to be taken with various constraints and about watering fields, can both be treated using ideas from graph theory.

## Where angels fear to tread?

Angel Treading Company has a number of branches all over Malgudi and there are a number of tracks that already exist. Now, the company wants to use all its existing tracks in such a way that it can get to any branch from any other branch, and the cost is minimized. So, it is vital to know the following: *How many routes must the company operate in order to serve all the sections without having more than one route on any section?*

To understand the problem, let us look at a simple situation to begin with. Suppose the branches are at $A$, $B$, $C$, $D$ and $P$, and the tracks are as in Figure 1.

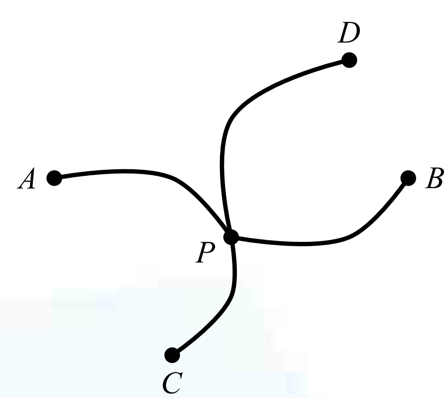


Figure 1

One route could go from $A$ to $B$ via $P$; another could go from $C$ to $D$, again via $P$. It is clear that these two routes suffice and it is also clear that two is the least number solving the problem. A person wanting to go from $A$ to $C$ could then take the first route until $P$ and change over to the route to $C$ from $P$.

Of course, the above solution is not unique; for instance, one could have a route from $A$ to $C$ via $P$ and another from $B$ to $D$ via $P$.

*Keywords: Networks, routes, vertices, odd, even, edges, faces, Euler's formula*

Let us look at another network as in Figure 2 which is slightly more complicated than the previous one.

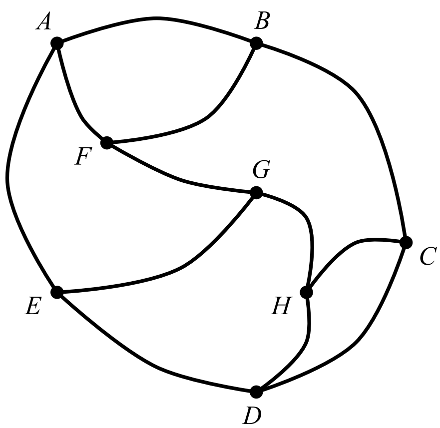


Figure 2

One route could be from *A* and run cyclically around *B, C, D, E* and back to *A*. Another could run from *A* to *F, G, H* and then to *D*. Three other routes *BF, EG, CH* would be needed, making the number of routes five in all. But, as we can see, we could combine the first two routes to make a single route and, therefore, four routes suffice for this network. We will prove in a moment that four is, indeed, the least number of routes needed.

The essential thing in the problem is to consider where the ends of the various routes must lie. Wherever a section of the track has a *free end*, as at *A, B, C, D* in Figure 1, there must be a start or end of a route. Since in Figure 1 there are four free ends, and since each route can have at the most two ends (a closed route has only one), clearly there must be at least two routes between the four free ends. By means of a single consideration, we have obtained the same result which we could obtain earlier only by considering all possible routes!

Let's look again at Figure 2 now. There are no free ends but there are junctions like *A* where three sections come together. At such a place, at least one route must start or end! Why? The reason is that any route passing through *A* has to use one section of the track while coming to *A* and another section while leaving *A*. So, the section of the track is left unpaired with any other section and has to be the start or end of a route. Of course, it might be that all three sections might be where routes start or end. That is why we said that there is at least one route ending or starting at *A*. In Figure 2 there are eight places of this kind, so there must be at least four routes; and as we saw, four routes will suffice.

As a final example, let us look at the network in Figure 3; there are five junctions of order 3 (i.e., where three sections come together) and one junction *F* of order 5. Again, obviously, there must be at least one route starting or ending at *F* since the order 5 of *F* is odd. So, there must be at least 6 route ends and therefore at least 3 routes are needed. Can you find 3 routes which suffice?

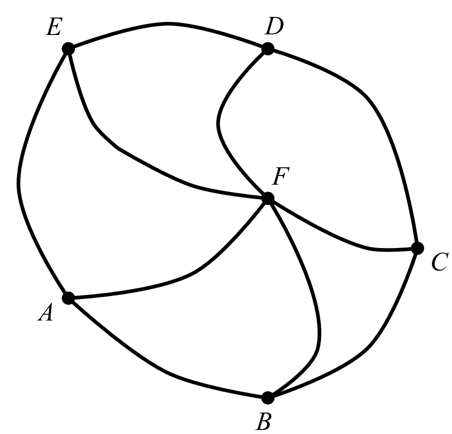


Figure 3

For any network, however complicated, we can count the number of junctions of odd order and divide by 2 to obtain the least possible number of routes. In the three examples, the number of junctions of odd order was always even, and it turned out that half this number was also sufficient.

We can see that for a system of routes to be optimal, the sections at each junction must be paired off, whenever possible. Why? Look at Figure 3 and look at the point *F*. If one route came from *C* to *F* and one from *D* to *F*, then both could be connected at *F* to form a single route and this would reduce the total number of routes. So, the conclusion from this discussion is: *In order for a system to be optimal,* *the sections at each junction must be paired off, whenever possible, and no route must end at an even junction; and then, the total number of ends of routes will be the number of odd order junctions, and the number of routes will be half the number of odd order junctions.*

One point still to be decided is whether a system that is optimal can contain a closed route. In Figure 2, we started with the closed route from *A* via *B, C, D, E* back to *A*, but then we connected it with the route from *A* through *F, G, H* to *D,* to make a single route from *A* to *D* (which is not closed). Such a reduction can be made when a closed route contains a junction of odd order. In fact, a similar reduction can be made when all junctions along the route have even order, as we show now. Let *A* be such a junction as in Figure 4 on a closed route, shown here in the shape of a figure of eight. Some other routes through *A* are also shown here (as dotted curves) and they might continue in any way.

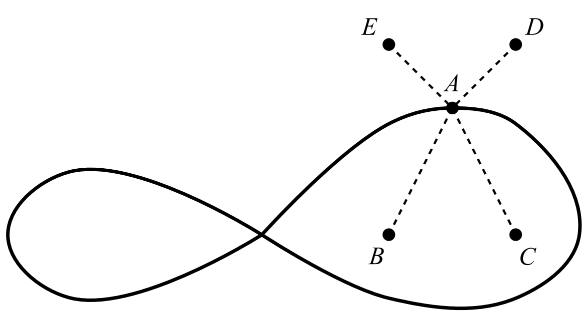


Figure 4

If the system is optimal, no route can end at *A*, so a route from *B* to *A* continues on through, say, to *E*. But then, we can combine these two routes by combining this route from *B* to *A* and along the closed route through *A* and then from *A* to *E*. This reduces the number of routes again. In this way, if we keep reducing closed routes with only even junctions, we will end up at some stage with a closed route with an odd junction and then the next reduction will yield a route that is not closed. Otherwise, *all* junctions in the original network must have been even, and then we can reduce the system to a single closed route.

Summing up our discussion, if a system is optimal, then:

a) Routes start or end only at junctions of odd order.

b) There will be a closed route only if all junctions in the original network are of even order, and then a single closed route will traverse the entire network.

c) The number of junctions of odd order is equal to the number of ends of routes, and is, therefore, an even number.

d) The minimum number of routes is half the number of junctions of odd order, except in the case where all junctions are of even order, when the minimum is one (closed) route.

**Euler's Formula**

Let us look at a map of fields and dikes (see Figure 5). Any two adjacent fields have a unique dike separating them. Think of the outside as being covered with water. We want to break the dikes one after another until all the fields are under water. (We may also think of this action as "opening the gate".) Suppose there are *f* fields to start with, *e* dikes and *v* vertices (or corners).

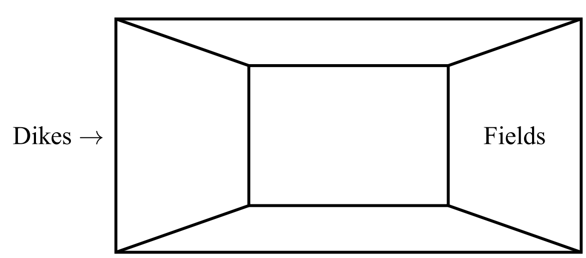


Figure 5. Here $f = 5$, $e = 12$ and $v = 8$

As you can see, it is not necessary to break *all* the dikes in order to water the fields. Any dike that already has water on both sides of it can certainly be left unbroken. If we break dikes that have water only on one side, then at each step we shall destroy one dike and flood one more field. Since this process can be carried out until all the fields have been flooded, and since we shall finally have flooded exactly *f* fields, we would have destroyed exactly *f* dikes at the end.

We want to count the number $e - f$ (of dikes left unbroken) in another way.

**One can walk dry-footed along the dikes from any vertex to any other vertex.** Before any dikes were broken, this could certainly have been done. Suppose in the course of flooding the fields, the destruction of some dike $AB$ (as in Figure 6a) would cut the system into two separate islands. If $AB$ were destroyed, it would be impossible to walk along dikes from $A$ to $B$. This means that water would completely surround each of the two islands. This means that water must have been on *both* sides of $AB$ before it is destroyed, and we stated that such a dike should not be destroyed. This shows that we can indeed walk along the dikes from any vertex to any other:

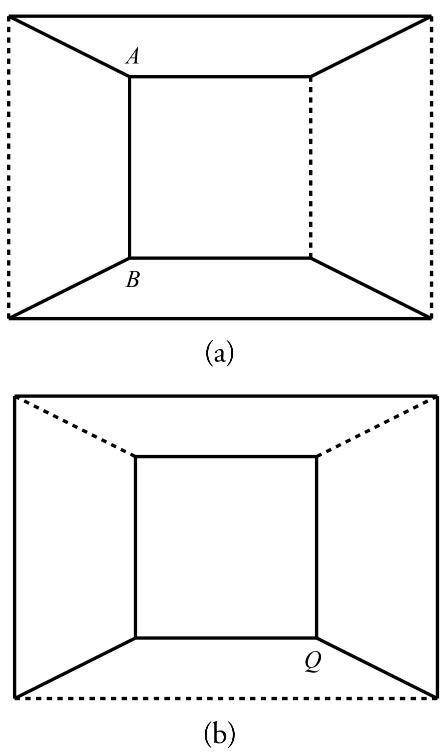


Figure 6. One can walk from any vertex to any other vertex, using the dikes

**There is exactly one path going along the dikes from one vertex to another.** If there were two paths from $P$ to $Q$, they would surround some area (see Figure 6b). The ring of undestroyed dikes surrounding this area will keep the area dry, contrary to the fact that all the fields have been flooded.

From these observations, we see that if we fix any starting point $P$, there is a unique undestroyed dike ending at any vertex (except $P$), and conversely, there is a unique end point for each edge.

**To summarize: *There are as many undestroyed dikes as there are end points of paths.***

Since the latter number is $v - 1$ (as $P$ is not an end point), we have $e - f = v - 1$.

This is called ***Euler's formula***. To state it in another form, look at a map with $F$ faces, $E$ edges and $V$ vertices. Then $V - E + F = 2$. (In our case, $F = f + 1$ since the water outside the fields is also a face.)

Euler's formula is a result of great power; it can be used to prove that *every map can be coloured with five colours*. What this means is that adjoining faces must have different colours (in a map) and five colours are sufficient to colour any map. Actually, four colours suffice but this is a very deep result proved using methods of topology.

**B. SURY** got his PhD in Mathematics from TIFR (Mumbai). He has been with the Indian Statistical Institute (Bangalore) since 1999. He has been interested in expository writing at school and college level and in interacting with mathematically talented students. He is the national coordinator for the Math Olympiad in Karnataka and a member of the editorial committee of the newsletter of the Ramanujan Mathematical Society. His research interests are in algebra and number theory. Mathematical limericks are an abiding interest. He may be contacted at sury@isibang.ac.in. His professional webpage is www.isibang.ac.in/~sury.