Features

Sums of Cubes

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We have all heard the taxicab story featuring GH Hardy and S Ramanujan, but we may not know that there are some wonderful problems dealing with sums of cubes which are currently yet to be solved. We describe some recent activity in this area.

Most of us have heard the taxicab story featuring GH Hardy and S Ramanujan and the number 1729. The story [1] concludes with Ramanujan telling Hardy that 1729 is "the smallest number expressible as the sum of two [positive] cubes in two different ways." As a result of this curious episode, numbers with such a property have come to be known as *taxicab numbers*. We have encountered these numbers in an earlier article [2] in AtRiA.

This article deals not with taxicab numbers but with another extremely interesting problem dealing with sums of cubes.

Sums of two cubes

To start with, we ask: *Which positive integers are sums of two cubes?* We must specify at the start whether we are permitted to use cubes of negative integers. We shall opt to do so. So the question we ask is:

> *Which positive integers n can be written in the form $n = a^3 + b^3$ where a, b are integers (which could be positive or negative)?*

Note that we could have opted to use only cubes of non-negative integers. That then becomes another problem, distinct from this one.

For the rest of this article, we shall consistently permit the use of cubes of negative numbers.

There are surely many more numbers which are sums of two cubes than numbers which are cubes. How many more? What can be said about these numbers? Let S represent the set of all positive integers n which can be written in the form $n = a^3 + b^3$ where a, b are integers, i.e.,

$$S = \{n \,:\, n = a^3 + b^3,\ a \in \mathbb{Z},\ b \in \mathbb{Z}\}.$$

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Note that S contains all the cubes. We display below the first 100 numbers in S (the table has been generated using computer software).

1	2	7	8	9	16	19	26	27	28
35	37	54	56	61	63	64	65	72	91
98	117	124	125	126	127	128	133	152	169
189	208	215	216	217	218	224	243	250	271
279	280	296	316	331	335	341	342	343	344
351	370	386	387	397	407	432	448	468	469
485	488	504	511	512	513	520	539	547	559
576	602	604	631	637	657	665	686	702	721
728	729	730	737	756	784	793	817	819	854
855	866	875	919	936	945	973	988	992	999

Inputs from modular arithmetic. In exploring the structure of a set of integers generated through any arithmetical procedure, it often helps to examine the set through the lens of modular arithmetic. We shall do the same with the set S.

Consider the possible remainders left when the cubes are divided by various natural numbers. On division by 2, remainders of 0 and 1 are possible; no great surprise here! On division by 3, remainders of 0, 1 and 2 are possible; once again, no surprise here. On division by 4, remainders of 0, 1 and 3 are possible, but not a remainder of 2. Continuing, we obtain the result shown in the table below.

Modulus	Remainders	Non-remainders
2	0, 1	
3	0, 1, 2	
4	0, 1, 3	2
5	0, 1, 2, 3, 4	
6	0, 1, 2, 3, 4, 5	
7	0, 1, 6	2, 3, 4, 5
8	0, 1, 3, 5, 7	2, 4, 6
9	0, 1, 8	2, 3, 4, 5, 6, 7

We see that the first really interesting cases are when the moduli are 7 and 9, as there are more non-remainders than remainders in both these cases. This permits the use of these two moduli for making useful characterisations. Let us now see how to make use of these observations.

In the subsequent analysis, we shall use only the modulus 9. The fact that the only remainders possible are $0, 1, 8$ allows us to state the following.

Theorem 1. Every cube is of one of the following forms: $9k$, $9k \pm 1$.

This immediately implies the following two corollaries. Recall that S represents the set of all positive integers n which can be written in the form $n = a^3 + b^3$ where a, b are integers.

Corollary 1. *Every number in S is of one of the following forms:* $9k$, $9k \pm 1$, $9k \pm 2$.

This may be stated in its contrapositive form as follows:

Corollary 2. *If a number is of the form* $9k \pm 3$ *or* $9k \pm 4$, *then it does not belong to S.*

The condition in the above corollary does not eliminate sufficiently many numbers from membership in S. We need to look for better characterisations of S, but these are not readily forthcoming. We may contrast this with the situation for squares, when we have an extremely compact characterisation available, namely: *A number n is expressible as a sum of two squares if and only if, in the prime factorisation of n, all prime factors of the form* $4k + 3$ *occur with even exponent.*

For the sum-of-two cubes problem, though characterisations are available, they are quite involved. For a recent result in this area, see [3].

The prime numbers in S. The prime numbers among the first 100 numbers in S are the following:

$$2,\ 7,\ 19,\ 37,\ 61,\ 127,\ 271,\ 331,\ 397,\ 547,\ 631,\ 919.$$

If we discard the very first number (the prime number 2), then a very curious pattern is noticed about all the remaining numbers. Namely, *they are all of the form* $9k + 1$ *or* $9k - 2$ *for some integer k*. It is worth asking whether this is a genuine pattern, i.e., true for all the odd primes in S, or a misleading pattern that persists only among the first few numbers in S. If it is true, it would imply that numbers in S which exceed 2 and are of the form $9k - 1$ or $9k + 2$ are all composite. So is this strange pattern genuine or not? The answer to this puzzle is not known.

But we have mentioned this observation only in passing, as a by-the-way. The central focus of this article is the sums-of-three-cubes problem, which we now discuss.

Sums of three cubes

We move to a consideration of numbers which can be written as the sum of three cubes. Let T represent the set of all natural numbers which are either cubes or sums of two cubes or sums of three cubes, i.e., sums of three or fewer cubes:

$$T = \left\{ n \ : \ n = a^3 + b^3 + c^3, \ a \in \mathbb{Z}, \ b \in \mathbb{Z}, \ c \in \mathbb{Z} \right\}.$$

Since every cube is of one of the forms $9k$, $9k \pm 1$, it follows that a sum of three or fewer cubes must be of one of the following forms: $9k$, $9k \pm 1$, $9k \pm 2$, $9k \pm 3$. We therefore have the following result:

Theorem 2. If a number is of the form $9k \pm 4$, then it does not belong to T.

It is very easy to generate elements of T, as many as we may want, simply by giving all possible integer values to a, b, c in some specified range (and with $|a| \leq |b| \leq |c|$ to avoid duplication of elements) and then computing the value of $a^3 + b^3 + c^3$.

But verifying whether a given number belongs to T (or not) is far more difficult. To get a glimpse of the difficulty involved, consider the following. The expressions for 99, 98 and 97 are easily found, as they involve relatively small numbers:

$$\begin{aligned} 99 &= 2^3 + 3^3 + 4^3, \\ 98 &= 0^3 + (-3)^3 + 5^3, \\ 97 &= (-1)^3 + (-3)^3 + 5^3. \end{aligned}$$

Similarly we have these expressions for 91, 92 and 93:

$$\begin{aligned} 91 &= 0^3 + 3^3 + 4^3, \\ 92 &= 1^3 + 3^3 + 4^3, \\ 93 &= (-5)^3 + (-5)^3 + 7^3. \end{aligned}$$

There are, of course, no such expressions for 94 and 95, as these are of the forbidden forms $9k \pm 4$. But for 96, we have all of a sudden:

$$96 = 10853^3 + 13139^3 + (-15250)^3.$$

And for 75, we have the following:

$$75 = 435203083^3 + (-435203231)^3 + 4381159^3.$$

To discover such relations, one clearly needs extremely powerful computational facilities. The complexities seem formidable.

At this stage, the following question poses itself quite naturally:

*If a number n is **not** of the form $9k \pm 4$, then does n belong to T?*

Stated in another (equivalent) form:

Are numbers that do not belong to T all of the form $9k \pm 4$?

In short, is the converse of Theorem 2 true?

Offhand, there does not seem any reason for supposing that it is true (or that it is false). But computational evidence seems to suggest otherwise! *The evidence overwhelmingly suggests that every number not of the form $9k \pm 4$ belongs to T.*

It is interesting to note how this conjecture evolved. Over the decades, mathematicians tried to express various numbers as sums of three cubes (of positive or negative integers), making use of powerful computational resources. By 1960, the only numbers less than 100 which had not yet been expressed in the required form were

$$30,\ 33,\ 39,\ 42,\ 52,\ 74,\ 75,\ 80,\ 84,\ 87,\ 91,\ 96.$$

Then in the 1960s the following relations were discovered:

$$\begin{aligned} 87 &= 4271^3 + (-4126)^3 + (-1972)^3, \\ 96 &= 13139^3 + (-15250)^3 + 10853^3, \\ 91 &= 83538^3 + (-67134)^3 + (-65453)^3, \\ 80 &= 103532^3 + (-112969)^3 + 69241^3. \end{aligned}$$

In the 1990s, the following relations were discovered:

$$39 = 134476^3 + (-159380)^3 + 117367^3$$
$$75 = 435203083^3 + (-435203231)^3 + 4381159^3$$
$$84 = 41639611^3 + (-41531726)^3 + (-8241191)^3$$

In the first decade of the 2000s, the following were discovered (the numbers keep getting bigger and bigger!):

$$30 = 2220422932^3 + (-2218888517)^3 + (-283059965)^3,$$
$$52 = 23961292454^3 + (-61922712865)^3 + 60702901317^3,$$
$$74 = 66229832190556^3 + (-284650292555885)^3 + 283450105697727^3.$$

By 2019, the only numbers below 100 which were not of the form $9k \pm 4$ and had not yet been expressed in the required form were 33 and 42. Then in mid-2019, the following mind-boggling relations were discovered:

$$33 = 8866128975287528^3 + (-8778405442862239)^3 + (-2736111468807040)^3,$$
$$42 = 80435758145817515^3 + (-80538738812075974)^3 + 12602123297335631^3.$$

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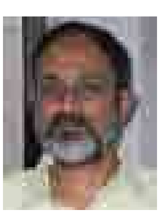

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For those who find it hard to imagine how such relations could ever have been found, here is an account of the story; it is excerpted from [7].

Mathematicians Solve '42' Problem With Planetary Supercomputer

MICHELLE STARR, 9 SEP 2019

Mathematicians have finally figured out the three cubed numbers that add up to 42*. This has settled a problem that has been pondered for 65 years …The problem, set in 1954, is …:* $x^3 + y^3 + z^3 = k$*. Here k is each of the numbers from* 1 *to* 100*; the question is, what are x, y and z?*

Over the decades, solutions were found for the easier numbers. In 2000, mathematician Noam Elkies of Harvard University published an algorithm to help find the harder ones.

This year, just the two most difficult ones remained: 33 *and* 42*.*

After watching a YouTube video [8] about the problem with 33 *on the popular maths channel Numberphile, mathematician Andrew Booker from the University of Bristol in the UK was inspired to write a new algorithm. He ran this through a powerful supercomputer at the university's Advanced Computing Research Centre, and got the solution for* 33 *after just three weeks.*

So, we were left with the hardest one of them all: 42*. This proved a much more obstinate problem, so Booker enlisted the aid of fellow MIT mathematician Andrew Sutherland, an expert in massively parallel computation.*

As you already know from the headline of this article, they figured it out. They also did a fun reveal of their success: according to The Aperiodical, both mathematicians quietly changed their personal websites to the solution, and named the pages "Life, the Universe, and Everything," a fitting nod to Douglas Adams.

Of course, it wasn't simple. The pair had to go large, so they enlisted the aid of the ***Charity Engine****, an initiative that spans the globe, harnessing unused computing power from over 500,000 home PCs to act as a sort of 'planetary supercomputer.' It took over a million hours of computing time, but the two mathematicians found their solution. …*

"I feel relieved," Booker said. "In this game, it's impossible to be sure that you'll find something. It's a bit like trying to predict earthquakes …So, we might find what we're looking for with a few months of searching, or it might be that the solution isn't found for another century."

Is that it, then? Well …no. That's just 1 *to* 100 *covered. Go up an order of magnitude to* 1000*, and there are still plenty of numbers to solve:* 114, 165, 390, 579, 627, 633, 732, 906, 921 *and* 975 *are all awaiting a solution to the sum of three cubes.*

Got any ideas?

We'll leave that question for the reader …