Hara's Triangle and Triple

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Even though the world of numbers and the world of geometry appear to be separate, there are many surprising connections between them. In Euclid's *Elements*, one finds many pretty connections mentioned that connect these two worlds.

A well-known connection is this: if a, b, c are three positive numbers, then a triangle with sides a, b, c can be constructed if and only if the sum of any two of the numbers is greater than the third number, i.e., if and only if $a + b > c$, $b + c > a$ and $c + a > b$.

Another well-known connection: when the sum of the squares of two of the numbers equals the square of the third number, then the triangle is right-angled. Thus, if $a^2 + b^2 = c^2$ for a triangle with sides a, b, c, then the angle opposite side c is a right angle.

While thinking about this, I wondered about the problem of finding solutions in positive integers to the equation $a^n + b^n = c^n$ where n is a negative integer. For example:

- If $n = -1$, then the question becomes: for what positive integers a, b, c is it true that $1/a + 1/b = 1/c$?
- If $n = -2$, then the question becomes: for what positive integers a, b, c is it true that $1/a^2 + 1/b^2 = 1/c^2$?

And so on.

Now let us look more closely at the above two cases.

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The case $n = -1$. Here the equation is $1/a + 1/b = 1/c$. Without much explanation, I am just giving solutions to the above equation because I want to focus on the case $n = -2$ as it gives rise to some unexpected surprises in geometry. Here are some solutions for $n = -1$:

$$(a, b, c) = (2, 2, 1),\ (3, 6, 2),\ (4, 12, 3),\ (5, 20, 4),\ (6, 30, 5),\ (7, 42, 6),\ (8, 56, 7),\ \ldots$$

The case $n = -2$. For $n = -2$ the equation can be written as $1/a^2 + 1/b^2 = 1/c^2$.

Before searching for the solutions, let me introduce the connection with geometry.

We know that in any triangle, if the sum of the squares of two sides is equal to the square of the third side, then the triangle is right-angled.

Instead of only considering relationships between the sides, what if we bring in other elements of the triangle, such as its altitudes, its medians and so on?

Imagine a triangle in which *the sum of the squares of two of its altitudes is equal to the square of the third altitude*. Specifically, in a triangle with sides a, b, c, let h_a, h_b, h_c denote the three altitudes (with h_a opposite side a and so on). We now ask: *For what positive integers a, b, c is it true that*

$$h_a^2 + h_b^2 = h_c^2\,?$$

To see the significance of this question, consider the area Δ of triangle ABC:

$$\Delta = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c.$$

From these relationships, we see that

$$h_a = \frac{2\Delta}{a}, \quad h_b = \frac{2\Delta}{b}, \quad h_c = \frac{2\Delta}{c}.$$

It follows from this that if

$$h_a^2 + h_b^2 = h_c^2,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2},$$

and conversely. This means that if we are able to construct a triangle whose sides are a, b, c such that $1/a^2 + 1/b^2 = 1/c^2$, then in that triangle the sum of the squares of the altitudes on sides a, b is equal to the square of the altitude on side c. And the converse is also true.

If such a triangle exists, I call it 'Hara's Triangle' and I call such a triple (a, b, c) 'Hara's Triple'.

Now let us look for positive integral solutions of the equation

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}. \tag{1}$$

This reminds us of a Pythagorean triple (x, y, z), i.e., which satisfies the relation

$$x^2 + y^2 = z^2.$$

A general formula which yields such triples is

$$x = 2m, \quad y = m^2 - 1, \quad z = m^2 + 1. \tag{2}$$

So the triple (a, b, c) will satisfy (1) if we can ensure that

$$a : b : c = \frac{1}{2m} : \frac{1}{m^2 - 1} : \frac{1}{m^2 + 1}. \tag{3}$$

A simple way to ensure this is to multiply through by $2m \cdot (m^2-1) \cdot (m^2+1)$ and take

$$a = (m^2-1)(m^2+1), \quad b = 2m(m^2+1), \quad c = 2m(m^2-1). \tag{4}$$

We see that for any integer $m > 1$, the integers a, b, c defined by $a = (m^2-1)(m^2+1)$, $b = 2m(m^2+1)$, $c = 2m(m^2-1)$ satisfy the equation

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}.$$

This means that for any $m > 1$, the triangle whose sides are $2m(m^2+1)$, $(m^2-1)(m^2+1)$ and $2m(m^2-1)$ has the property that the sum of the squares of two altitudes is equal to the square of the third altitude.

Here is a list of some Hara's Triples as generated by the above formula:

m	$a = (m^2-1)(m^2+1)$	$b = 2m(m^2+1)$	$c = 2m(m^2-1)$
2	15	20	12
3	80	60	48
4	255	136	120
5	624	260	240
6	1295	444	420
7	2400	700	672
8	4095	1040	1008
9	6560	1476	1440
10	9999	2020	1980

If we divide out the gcd of a, b, c from a, b, c (thereby obtaining a triangle similar to the original one, so that it has the same geometrical properties), we obtain the following triples a', b', c':

m	a'	b'	c'
2	15	20	12
3	20	15	12
4	255	136	120
5	156	65	60
6	1295	444	420
7	600	175	168
8	4095	1040	1008
9	1640	369	360
10	9999	2020	1980

Closing remark. The above exploration raises many other related questions in our minds. For example, instead of altitudes, we could ask what happens if the same condition is applied to medians or angular bisectors. And so on.

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