ClassRoom

Conversations on Greatest Common Divisor

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Today I am going to share with you an interesting conversation between me and my cousin Apoorvi, who is a curious student of class 9. This conversation began after she saw me teaching Greatest Common Divisor to Priyanka. Priyanka lives in my neighborhood and she is a student of class four. Sometimes she visits my home for help in mathematics. One day after helping Priyanka with her homework, I started talking to Apoorvi.

Me: Do you know how to calculate Greatest Common Divisor (GCD)?

Apoorvi: Oh, come on Bhaiya! Of course, I know!! I know five methods of finding it.

Me: Five methods? Even I don't know there were five methods of finding GCD.

Apoorvi: Ok!! Let me explain them to you one by one. We can calculate GCD by first writing the given numbers in the prime factor form. For example, $60 = 2 \times 2 \times 3 \times 5$ and $24 = 2 \times 2 \times 2 \times 3$. After writing them in their prime factor form, the multiplication of common primes i.e. $2 \times 2 \times 3$ is GCD of 60 and 24.

Me: This is the first method of which I am aware. We call this the prime factorization method. Can you tell me why this method works? I mean why we could find GCD by following the steps you explain?

Apoorvi: It's simple and the reason is hidden in the name- GCD. The Greatest Common Divisor of two (or more) numbers is the greatest number that divides both (or all) of them. In the prime factorization method, each number is a unique combination of prime numbers (each one of them divides the number) and all (and only all) the common factors are multiplied to get the GCD by the prime factorization method.

Me: Fair enough!! Tell me another method.

Apoorvi: Suppose we have to find the GCD of two numbers 60 and 24. For this, we first divide 60 by 24 and find the remainder, which is 12. Again 24 is divided by the remainder 12 and we get the remainder 0. The process ends and the last divisor 12 is the GCD of 60 and 24.

Keywords: factor, prime, common factor, GCD/HCF, multiple, algorithm, reasoning why

Me: Fine. This is the long division method of finding GCD, but you have to tell me three more methods, yet.

Apoorvi: Yes! Yes! Have patience! I am going to tell you all the five methods. We can also calculate GCD by square tiles. (Drew Figure 1 and explained).

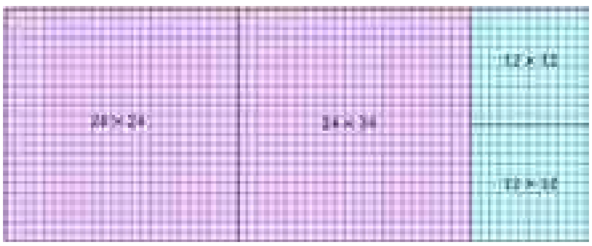


Figure 1: Calculating GCD using square tiles.

To find the GCD of 60 and 24, draw a rectangle of dimension 60 × 24. Now make as many squares of dimension 24 × 24 as possible. What remains is a rectangle of dimension 12 × 24. Again, we make as many 12 × 12 squares as possible. Is there any rectangle left? The answer is no and therefore 12 is GCD of 60 and 24.

(I was surprised that Apoorvi was perceiving this as a completely new method and was unable to see the interrelationships between methods 2 and 3. But instead of discussing the relationship between these two methods, I chose to listen to the next two methods. So, I asked her for the 4th method.)

Me: Interesting!! Tell me the next method of finding GCD.

Apoorvi: We can find GCD by strips also. (Again, she drew Figure 2).

To find the GCD of 60 and 24 we first draw a strip of length 60 and then draw as many times as possible, strips of length 24 starting from one end. We were able to fit two strips of length 24 and on checking the length of the remaining strip, we found it to be 12. Then we draw strips of length 12 on the strip of length 24 (starting again from one end, see Figure 3.). This time, there is nothing left after drawing two strips of length 12. Therefore, the GCD of 60 and 24 is 12.

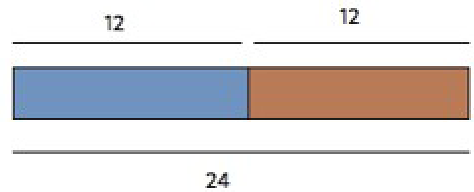


Figure 3: Calculating GCD using strips

Me: I want you to think on some other aspect of the methods shared by you but before that tell me your fifth method.

Apoorvi: I learnt about this method recently. The name of this method is Euclid division algorithm. My teacher told me that this method was first explained by Euclid in his book 'Elements'. Using this method, we can find GCD of 60 and 24 by successive /repeated division till we arrive at zero as remainder, as follows –

$60 = 2 \times 24 + 12$

The remainder 12 is between 0 and 24

$24 = 2 \times 12 + 0$ There is no remainder

Here we get remainder 0, so 12 is GCD of 24 and 60.

Me: Here are two questions for you.

1. Can you explain this method in your own words?
2. Why is 2 not the GCD of 60 and 24?

Apoorvi: (A little puzzled and in a complaining tone) Bhaiya, why do you always ask me to say the methods in words? The method says that for any pair of positive integers a, b, with (say) $0 < b \le a$, we can write $a = q \times b + r$ where $0 \le r$

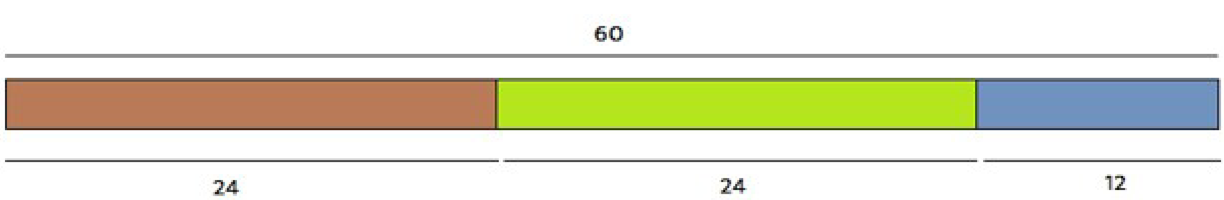


Figure 2: Calculating GCD using strips.

< b; here b is the divisor, q is the quotient, and r is the remainder; b and r are non-negative integers. If the remainder $r = 0$, then it means that a is a multiple of b, so b is itself the GCD of a and b. If $r > 0$, then nothing prevents us from dividing b by r and writing $b = r \times b_1 + r_1$ where $0 \leq r_1 < b_1$. If the remainder $r_1 > 0$, then we continue the division process until the remainder is zero, and then the last divisor is the GCD of a and b.

Me: Ok!! Apoorvi, I am happy that you know so many methods of finding the GCD. But I doubt if they are all distinct methods. So, I am giving you a few questions to think about.

1. *What is the basic argument in all these methods? Why do these methods work? I mean, how do these processes generate the GCD?*
2. *Is there any relationship among these methods?*

Apoorvi: No, No!! This is not fair! You know the answers. Instead of leaving me with your questions, tell me the answers. Otherwise, you know, your questions will keep bothering me until I find the explanations.

Me: Apoorvi, these questions are bothering me as well and right now, I do not have any well-articulated and thoughtful response to them. So, I suggest you also think about them as I will too.

Apoorvi: Ok!! I never thought about these. But I will soon tell you the answer of these questions.

This was my conversation with Apoorvi and I had forgotten about it. But the result of our conversation came after months. I was enjoying the evening, reading the novel 'A Certain Ambiguity', cup of tea at hand and slow Garhwali music in the background. Suddenly, Apoorvi entered the room, note book in hand. She seemed particularly happy. From the brightness in her eyes, it was evident that she wanted to share something exciting with me.

Apoorvi: (With excitement) Bhaiya! Bhaiya! I got it! I got it!

Me: (Little confused) I got it? What did you get, Apoorvi?

Apoorvi: Answers to your questions.

Me: What questions?

Apoorvi: Bhaiya! You remember our conversation on GCD which we had a few months back and at the end of which you left me with some questions?

Me: Conversation on GCD? I can't remember it. Give me some details so that I can recall.

Apoorvi: When I told you that there are five methods of finding GCD of any two numbers and described them to you one by one.

Me: Yes, Yes! Now I remember. I remember even the questions but sorry I did not find time to give thought to them. Well, you tell me.

Apoorvi: Ok, Bhaiya. Now I am going to tell you my findings. First, I found that they are not all distinct methods. The first method is different but the remaining four are based on the same mathematical argument. So, there is a clear relationship among the last four methods.

Me: Oh really! I am curious to know what the mathematical argument is.

Apoorvi: Before telling you the argument, I would like to tell you how I concluded what these methods are based on. I made the following table for finding the GCD which reveals the mathematical argument of methods for finding GCD.

In the above table, I took two arbitrary numbers and filled out the table. In the next line, I took the smaller of the arbitrary numbers and the remainder when the bigger number was divided by the smaller. I continued the same process until I got a remainder of 0. From the table I observed that the GCD by factorization is equal to the last divisor on successive division. This is the argument we applied in the last four methods.

Me: Can you elaborate, Apoorvi?

Apoorvi: Lets us see, one by one. In the second method we first divide 60 by 24 which means that we see how many 24s are there in 60 and find the remainder 12. Then again, we see how many 12s are there in 24 and considering the

Trial No	Numbers a and b ($a > b$)	Prime factorization of bigger number 'a'	Prime factorization of smaller number 'b'	GCD of a & b	Remainder when $a \div b$
1	60 and 24	$2 \times 2 \times 3 \times 5$	$2 \times 2 \times 2 \times 3$	12	$12 = 60 - 2(24)$
	Now taking 24 and 12	$2 \times 2 \times 2 \times 3$	$2 \times 2 \times 3$	12	$0 = 24 - 2(12)$
2	56 and 16	$2 \times 2 \times 2 \times 7$	$2 \times 2 \times 2 \times 2$	8	$8 = 56 - 3(16)$
	Now taking 16 and 8	$2 \times 2 \times 2 \times 2$	$2 \times 2 \times 2$	8	$0 = 16 - 2(8)$
3	165 and 65	$5 \times 3 \times 11$	5×13	5	$35 = 165 - 2(65)$
	Now taking 65 and 35	5×13	5×7	5	$30 = 65 - 1(35)$
	Now taking 35 and 30	5×7	$5 \times 2 \times 3$	5	$5 = 35 - 1(30)$
	Now taking 30 and 5	$5 \times 2 \times 3$	5	5	$0 = 30 - 6(5)$

GCD of 12 and 24 is the same as GCD of 24 and 60. In the rectangle method again we do the same thing that is how many 24s are there in 60. This is equivalent to dividing 60 by 24 and further division of 24 by the remainder 12. Similar process is applied the last two methods also.

Me: Brilliant! You rightly got the core of the argument. But are you sure about your argument?

Apoorvi: No Bhaiya, I am not sure because I have validated it with a few numbers only. I cannot claim that this result is valid for every single case. For this, we need to prove it. Will you help in proving this result?

Me: Ok! Before starting I want to make the declaration that here I am only talking about the set of whole numbers. And since your algebra is quite good I am going to use variables to represent the general case.

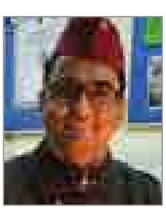
Pedagogical Notes: The table below has two columns. In the left hand column, we will use numbers to illustrate the argument. In the right hand column, we prove the result for the general case. We would advise that the general case is proved for students only when they are comfortable with the algebra. Of course, they must be made to understand that proof by example is not valid.

12 is the GCD of 60 and 24 Then: 1. 12 divides both 60 and 24 $60 = 12 \times 5$ $24 = 12 \times 2$ 2. 12 is the greatest common divisor i.e. 5 and 2 are co-prime, they have no common factors.	If, 'd' is the GCD of given two numbers a and b, then: 1. d divides both a and b $a = d \times \alpha$ $b = d \times \beta$ 2. d is the greatest common divisor i.e. α and β will be co-prime i.e. they will have no common factors.
For whole numbers, say, 12 and 17 $17 = 1 \times 12 + 5$; where $0 \leq 5 < 12$ Here 5 is the remainder when 12 divides 17	In general, for whole numbers a and b $a = q \times b + r$, where $0 \leq r < b$. Here r is the remainder when b divides a
$60 = 2(24) + 12$ So $12 = 60 - 2(24)$ $= (12 \times 5) - (2 \times 2 \times 12)$ $= 12(5 - (2 \times 2))$	Since $a = q \times b + r$ so $r = a - qb$ $= d\alpha - qd\beta$ $= d(\alpha - q\beta)$

So 12 divides both 60 and 24 and also 12	So d divides a and b and also r.
We have to now show that 12 is also the GCD of 24 and 12. We have seen that 12 divides both 24 and 12. Now we show that 12 is their greatest divisor.	We have to now show that d is the GCD of b and r. We have seen that d' is a divisor of both b and r. Now we show that d is their greatest divisor.
$24 = 12 \times 2$ $12 = 12 \times 1$ And 1 and 2 are co-prime So 12 is the GCD of 12 and 24.	$b = d\beta$ $r = d(\alpha - q\beta)$ Now d' will be the GCD of b and r if β and $(\alpha - q\beta)$ are co-prime
	On the contrary, let us assume β and $(\alpha - q\beta)$ are not coprime. Then we will have a number c which will divide both β and $(\alpha - q\beta)$. Symbolically $c \mid \beta$ and $c \mid (\alpha - q\beta)$.
	we can write $\alpha = \{q\beta + (\alpha - q\beta)\}$ Since c divides $q\beta$ and $(\alpha - q\beta)$ therefore c divides α.
	Thus, we can say that c divides both α and β which is a contradiction to the fact that α and β are co-prime. This implies that our assumption that number c divides both β and $(\alpha - q\beta)$ is wrong. This concludes that β and $(\alpha - q\beta)$ are co-prime. If β and $(\alpha - q\beta)$ are coprime then 'd' is the GCD of b and r too. We can continue like this as we divide successively and prove that the GCD of a and b is also the GCD of the successive remainders.

Apoorvi: Oh! Thank you Bhaiya! I got it. I never thought small concepts like GCD might have such relationships and insights.

Note from Author: The author would like to express his gratitude to Swati Sircar for encouraging him to write this article and helping him hone his writing skills.



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