ClassRoom

Impossible Triangles on Dot Sheets

SWATI SIRCAR

In the March 2019 issue of AtRiA, there were several proofs given explaining why it is not possible to draw an equilateral triangle on a rectangular dot sheet. Similarly, it is not possible to draw a right isosceles triangle (or a square) on an isometric dot sheet. We provide a proof of the latter.

Right Isosceles (or Square) on Isometric Dot Sheet

The rectangular dot sheet looks the same regardless of its orientation. But that is not the case for the isometric one. Usually, we take the minimum distance between two grid points as the unit distance. So, in one orientation the grid points along any horizontal line are unit distance apart but along any vertical line they are further apart (Figure 1). If we change the orientation, then the points along any vertical line become unit distance apart (Figure 2). Without loss of generality, we can orient the isometric dot sheet (or grid) so that on the y-axis, grid points are unit distance apart (see Figure 2).

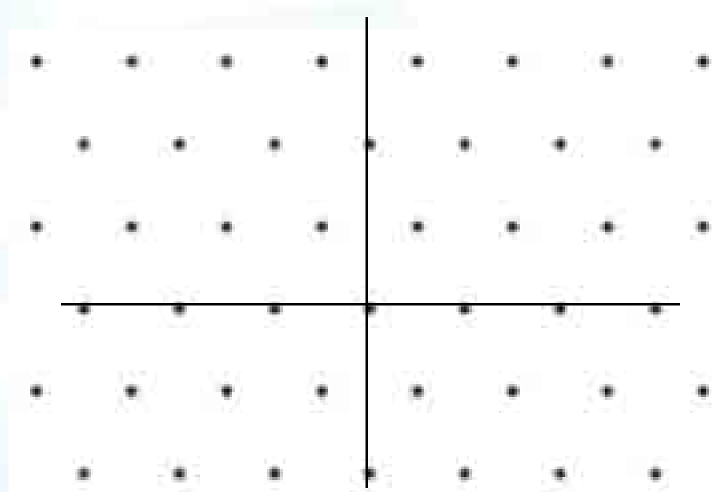

Figure 1

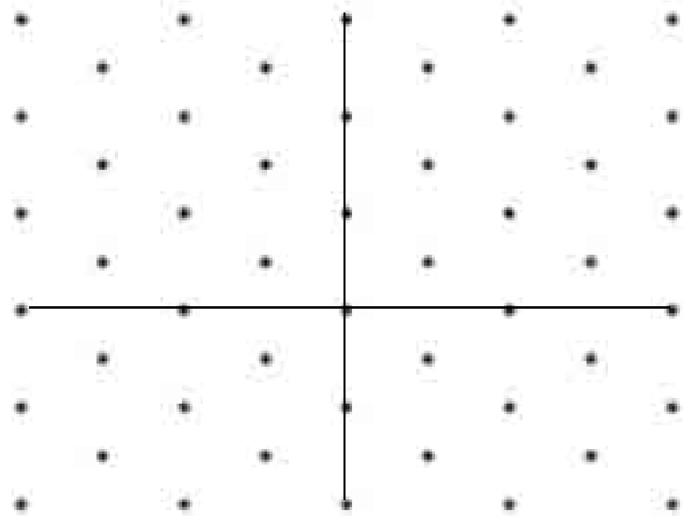

Figure 2

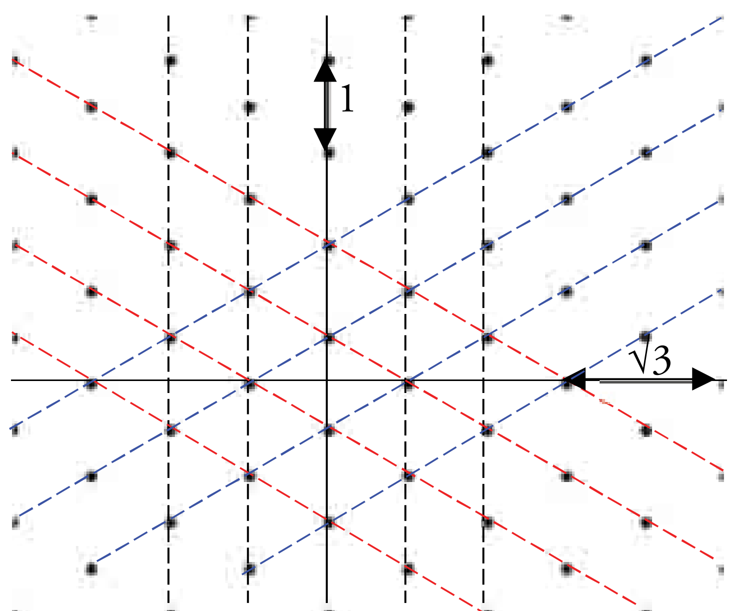

Figure 3

Note that there are 3 sets of parallel lines that we can get by joining the nearest points – the black lines parallel to the y-axis, the blue ones and the red ones (Figure 3). These three sets form equilateral triangles. So, the blue lines form 60° (and 120°) angles with the y-axis, and therefore 30° (and 150°) angles with the x-axis. Similarly, the red lines form 120° (and 60°) and 150° (and 30°) angles with the y- and the x-axis respectively. Therefore, the distance between two consecutive dots along the x-axis is twice the altitude of an equilateral triangle with unit sides i.e. $\sqrt{3}$ units.

So, as shown in Figure 4, the points along the x-axis then have x-coordinates $0, \pm\sqrt{3}, \pm2\sqrt{3}, \pm3\sqrt{3}, \ldots,$ i.e., their coordinates have the form $\left(m\sqrt{3}, 0\right)$ for some integer m. Similarly, points on any row with integer y-co-ordinates are of the form $\left(m\sqrt{3}, n\right)$ for some integers m and n.

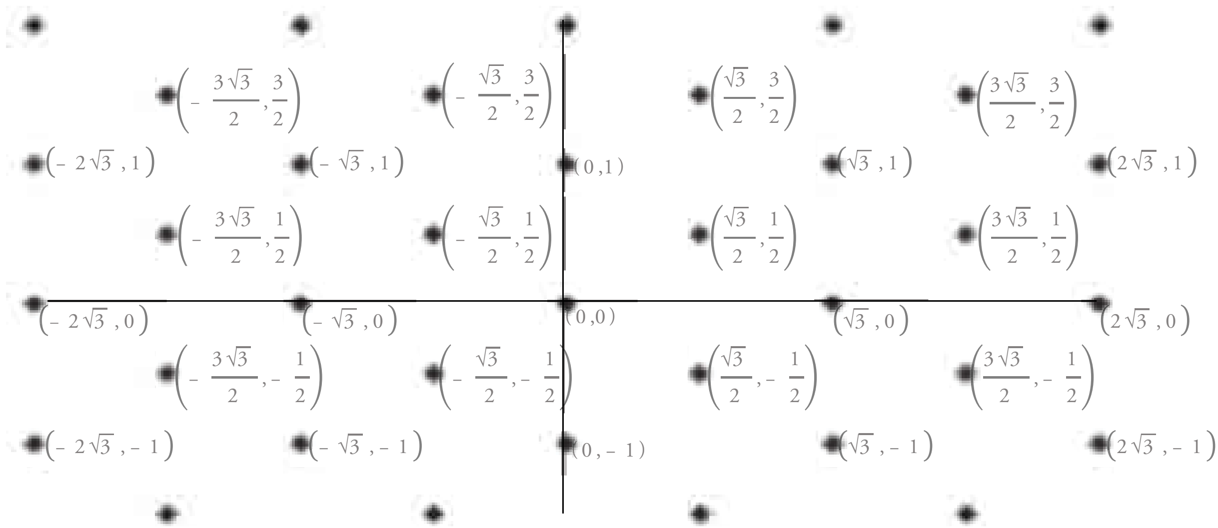

Figure 4

The points on the rows immediately above and below the x-axis have x-coordinates $\pm\frac{\sqrt{3}}{2}, \pm\frac{3\sqrt{3}}{2}, \pm\frac{5\sqrt{3}}{2} \ldots$, i.e., they are all odd multiples of $\frac{\sqrt{3}}{2}$. And the y-coordinates of these points are $\pm\frac{1}{2}$. This pattern continues for all rows with non-integer y-coordinate. For such rows, the points have coordinates $\left(\frac{(2m+1)\sqrt{3}}{2}, n+\frac{1}{2}\right)$ for some integers m and n. In short, the coordinates of points on an isometric grid are of the form $\left(\frac{m\sqrt{3}}{2}, \frac{n}{2}\right)$ where m, n are either both even or both odd.

Now, if we can draw a right isosceles triangle on such a grid, we may assume without loss of generality that the vertex of the triangle corresponding to the right angle coincides with the origin of the grid. (To accomplish this, we translate the triangle parallel to itself so that the vertex corresponding to the right angle coincides with the origin.) Let P and Q be the remaining two vertices. Then $OP = OQ$ and $OP \perp OQ$.

Now there are three possibilities considering the parity of the coordinates of P and Q; namely, their m, n values may be (i) both even, (ii) both odd, (iii) one even and the other odd. We consider each of these in turn, starting with (i).

Let $P = \left(m\sqrt{3}, \text{n}\right)$ and $Q = \left(r\sqrt{3}, s\right)$ where m, n, r, s are integers.

The product of the slopes of OP and OQ is -1, so

$$\frac{n}{m\sqrt{3}} \times \frac{s}{r\sqrt{3}} = -1, \quad \therefore s = -\frac{3mr}{n}. \tag{3}$$

And $OP^2 = OQ^2$, so

$$n^2 + 3m^2 = s^2 + 3r^2, \quad \therefore s^2 = n^2 + 3\left(m^2 - r^2\right). \tag{4}$$

Combining (3) and (4), we get:

$$n^2 + 3\left(m^2 - r^2\right) = \frac{9m^2r^2}{n^2}, \quad \therefore n^4 + 3\left(m^2 - r^2\right)n^2 - 9m^2r^2 = 0,$$
$$\therefore \left(n^2 + 3m^2\right)\left(n^2 - 3r^2\right) = 0, \therefore n = \pm r\sqrt{3}.$$

This is not possible since $\sqrt{3}$ is irrational.

For (ii), let $P = \left(\frac{(2m+1)\sqrt{3}}{2}, n + \frac{1}{2}\right)$, $Q = \left(\frac{(2r+1)\sqrt{3}}{2}, s + \frac{1}{2}\right)$. So, the slope equation becomes:

$$\frac{\left(n+\frac{1}{2}\right)\left(s+\frac{1}{2}\right)}{\frac{3}{4}\left(2m+1\right)\left(2r+1\right)} = -1, \therefore \left(2n+1\right)\left(2s+1\right) = -3\left(2m+1\right)\left(2r+1\right).$$

Note that this is similar to what we got in (i), but with the following changes:

$$m \to 2m+1, \quad r \to 2r+1, \quad n \to 2n+1, \quad s \to 2s+1.$$

Therefore, this reduces to $2n+1 = \pm\left(2r+1\right)\sqrt{3}$, i.e., an impossibility as earlier.

For (iii), let $P = \left(\frac{(2m+1)\sqrt{3}}{2}, n + \frac{1}{2}\right)$ and $Q = \left(r\sqrt{3}, s\right)$, without loss of generality. The slope equation now becomes

$$\frac{\left(n+\frac{1}{2}\right)s}{\frac{3}{2}\left(2m+1\right)r} = -1, \therefore s = -\frac{3\left(2m+1\right)r}{2n+1}.$$

This is again similar to (i), but with the changes $m \to 2m+1$ and $n \to 2n+1$. Consequently, we get $2n+1 = \pm r\sqrt{3}$, an impossibility as earlier.

We conclude that constructing a right isosceles triangle on an isometric grid is not possible.

SWATI SIRCAR is Assistant Professor at the School of Continuing Education and University Resource Centre of Azim Premji University. Mathematics is the second love of her life (first being drawing). She has a B.Stat-M.Stat from Indian Statistical Institute and an MS in mathematics from University of Washington, Seattle. She has been doing mathematics with children and teachers for more than 5 years and is deeply interested in anything hands on, origami in particular. She may be contacted at swati.sircar@apu.edu.in.