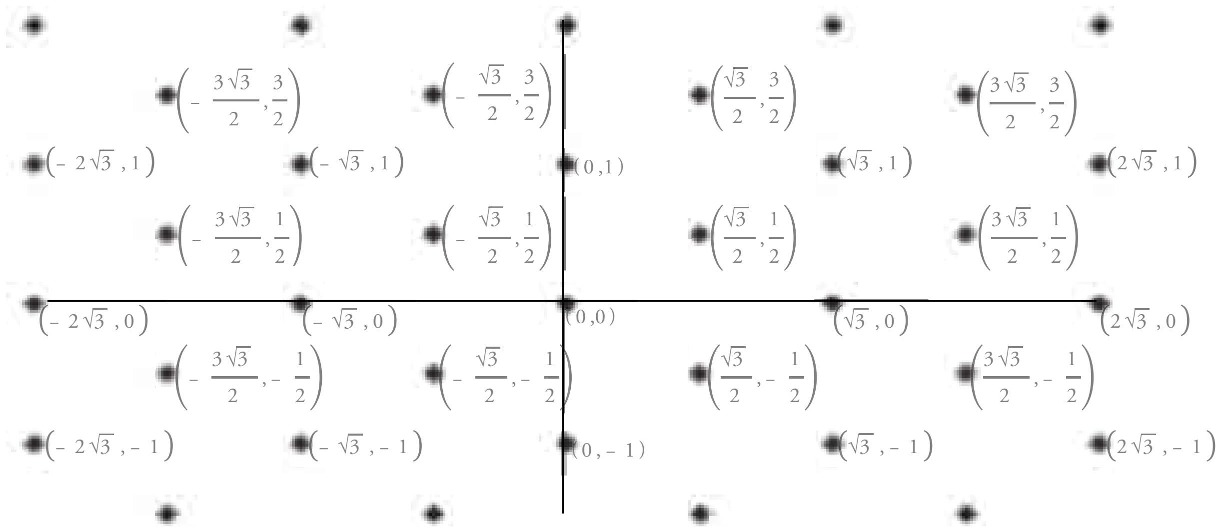

Figure 4

The points on the rows immediately above and below the x-axis have x-coordinates $\pm\frac{\sqrt{3}}{2}, \pm\frac{3\sqrt{3}}{2}, \pm\frac{5\sqrt{3}}{2} \ldots$, i.e., they are all odd multiples of $\frac{\sqrt{3}}{2}$. And the y-coordinates of these points are $\pm\frac{1}{2}$. This pattern continues for all rows with non-integer y-coordinate. For such rows, the points have coordinates $\left(\frac{(2m+1)\sqrt{3}}{2}, n+\frac{1}{2}\right)$ for some integers m and n. In short, the coordinates of points on an isometric grid are of the form $\left(\frac{m\sqrt{3}}{2}, \frac{n}{2}\right)$ where m, n are either both even or both odd.

Now, if we can draw a right isosceles triangle on such a grid, we may assume without loss of generality that the vertex of the triangle corresponding to the right angle coincides with the origin of the grid. (To accomplish this, we translate the triangle parallel to itself so that the vertex corresponding to the right angle coincides with the origin.) Let P and Q be the remaining two vertices. Then $OP = OQ$ and $OP \perp OQ$.

Now there are three possibilities considering the parity of the coordinates of P and Q; namely, their m, n values may be (i) both even, (ii) both odd, (iii) one even and the other odd. We consider each of these in turn, starting with (i).

Let $P = \left(m\sqrt{3}, \mathrm{n}\right)$ and $Q = \left(r\sqrt{3}, s\right)$ where m, n, r, s are integers.

The product of the slopes of OP and OQ is -1, so

$$\frac{n}{m\sqrt{3}} \times \frac{s}{r\sqrt{3}} = -1, \quad \therefore s = -\frac{3mr}{n}. \tag{3}$$

And $OP^2 = OQ^2$, so

$$n^2 + 3m^2 = s^2 + 3r^2, \quad \therefore s^2 = n^2 + 3\left(m^2 - r^2\right). \tag{4}$$

Combining (3) and (4), we get:

$$n^2 + 3\left(m^2 - r^2\right) = \frac{9m^2r^2}{n^2}, \quad \therefore n^4 + 3\left(m^2 - r^2\right)n^2 - 9m^2r^2 = 0,$$
$$\therefore \left(n^2 + 3m^2\right)\left(n^2 - 3r^2\right) = 0, \therefore n = \pm r\sqrt{3}.$$

This is not possible since $\sqrt{3}$ is irrational.

For (ii), let $P=\left(\frac{(2m+1)\sqrt{3}}{2}, n+\frac{1}{2}\right)$, $Q=\left(\frac{(2r+1)\sqrt{3}}{2}, s+\frac{1}{2}\right)$. So, the slope equation becomes:

$$\frac{\left(n+\frac{1}{2}\right)\left(s+\frac{1}{2}\right)}{\frac{3}{4}\left(2m+1\right)\left(2r+1\right)}=-1, \therefore (2n+1)(2s+1)=-3(2m+1)(2r+1).$$

Note that this is similar to what we got in (i), but with the following changes:

$$m\to 2m+1, \quad r\to 2r+1, \quad n\to 2n+1, \quad s\to 2s+1.$$

Therefore, this reduces to $2n+1=\pm(2r+1)\sqrt{3}$, i.e., an impossibility as earlier.

For (iii), let $P=\left(\frac{(2m+1)\sqrt{3}}{2}, n+\frac{1}{2}\right)$ and $Q=\left(r\sqrt{3}, s\right)$, without loss of generality. The slope equation now becomes

$$\frac{\left(n+\frac{1}{2}\right)s}{\frac{3}{2}\left(2m+1\right)r}=-1, \therefore s=-\frac{3(2m+1)r}{2n+1}.$$

This is again similar to (i), but with the changes $m\to 2m+1$ and $n\to 2n+1$. Consequently, we get $2n+1=\pm r\sqrt{3}$, an impossibility as earlier.

We conclude that constructing a right isosceles triangle on an isometric grid is not possible.

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ClassRoom

Triangle Centres – Barycentric Coordinates

A. RAMACHANDRAN

In an earlier article we had presented a way to characterize well known triangle centres – by their 'trilinear coordinates.' (Ref. 1) In this article we take up another system of characterising triangle centres, where each triangle centre is considered as the location of the centre of mass of a system of three point masses placed at the vertices of the triangle. The ratio of the masses then forms the 'barycentric coordinates' of the point in question. This approach was suggested by the German mathematician August Ferdinand Mobius in 1827.

Centroid

We first look at the centroid. This point is the centre of mass of a system of three equal masses placed at the vertices of the triangle, as discussed below. The centre of mass of the masses at B and C lies at the midpoint of BC, say D. So the centre of mass of all three must lie on (median) AD. Similarly we could say that the overall centre of mass should lie on the medians BE (E, midpoint of CA) and CF (F, midpoint of AB) as well. So, the overall centre of mass lies on the point of concurrence of the medians, a fact guaranteed to us by the converse of Ceva's theorem, since $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$. Hence the barycentric coordinates of the centroid are $1:1:1$.

We also note that the point of concurrence of the medians, the centroid G, divides AD in the ratio $2:1$, which is the inverse of the ratio of the mass at A to the combined masses at B and C.

Keywords: Coordinates, barycentric, centre of mass, centroid, ratio

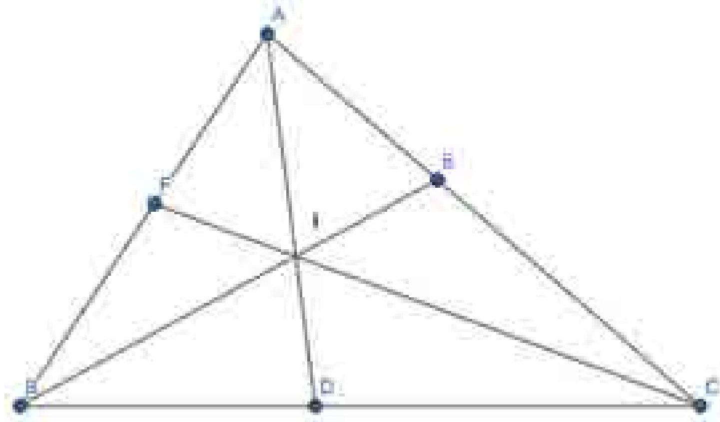

Figure 1

Incentre

We now take up the incentre. Let AD, BE, CF be the angle bisectors in Δ ABC (Figure 1). It is a well-known result that D divides BC in the ratio of the sides AB : AC or $c : b$. So if masses (proportional to) b and c were placed at points B and C, respectively, their centre of mass would lie at D. If now mass a were placed at A, then the centre of mass of all three would lie on the angle bisector AD. Similar arguments lead to the conclusion that the centre of mass of the system would lie on the other angle bisectors as well, or at their point of concurrence, the incentre I, the fact of concurrence guaranteed by the converse of Ceva's theorem, as $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = \frac{c}{b} \times \frac{a}{c} \times \frac{b}{a} = 1$. So the barycentric coordinates of the incentre are $a : b : c$, or sin A : sin B : sin C.

Note that in Δ ABD, BI bisects ∡B, and so AI : ID = AB : BD = $c : \frac{ac}{b+c} = (b+c) : a$. Thus, I divides AD in a ratio that is the inverse of the ratio of the mass at A to the combined masses at B and C.

Orthocentre

We now consider the orthocentre. We first look at an acute angled triangle, say Δ ABC, with altitudes AD, BE, CF, meeting at H (Figure 2).

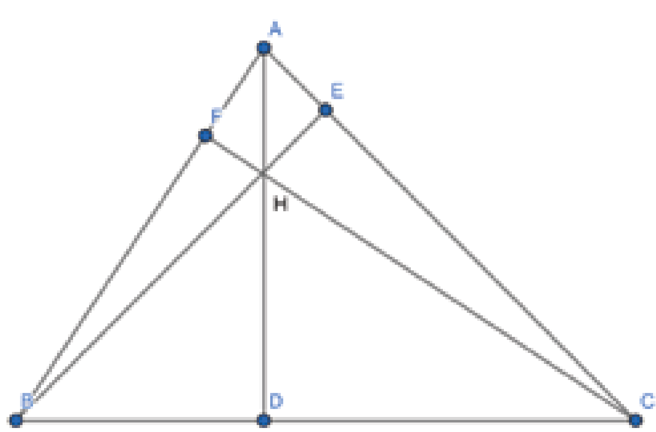

Figure 2

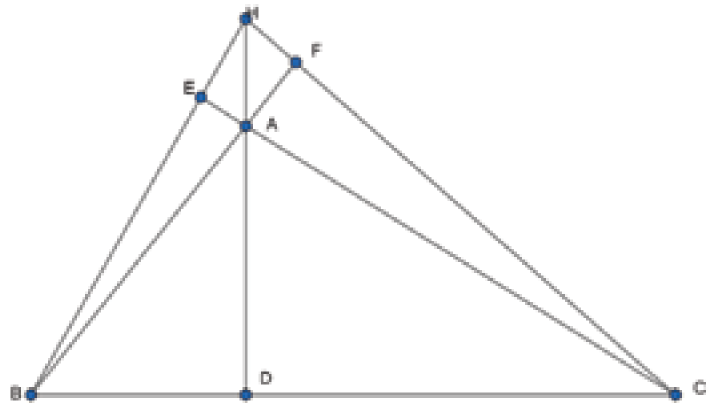

Figure 3

We have $\dfrac{BD}{DC} = \dfrac{AD/DC}{AD/BD} = \dfrac{\tan C}{\tan B}$. We could say that if masses (proportional to) tan B and tan C were placed at B and C, respectively, their centre of mass would lie at D. If now mass tan A were placed at A then the centre of mass of all three would lie on the altitude AD. By similar arguments we could say that the overall centre of mass lies on altitudes BE and CF too, or at their point of concurrence. Again, concurrence is guaranteed by the converse of Ceva's theorem. So the barycentric coordinates of the orthocentre are tan A : tan B : tan C.

Also note that $\dfrac{AD}{HD} = \dfrac{AD}{DC} \times \dfrac{DC}{HD} = \tan B\ \tan C$ and

$$\frac{AH}{HD} = \frac{AD}{HD} - 1 = \tan B\ \tan C\ - 1 = \frac{\tan B\ + \tan C}{\tan A}.$$

(The last step follows from the relation tan A tan B tan C = tan A + tan B + tan C, for A + B + C = 180° .)

Thus, H divides AD in a ratio that is the inverse of the ratio of the mass at A to the combined masses at B and C.

In the case of an obtuse angled triangle (see Figure 3), we have $\dfrac{CE}{EA} = \dfrac{CE/HE}{EA/HE} = \dfrac{\tan\ (180^\circ - A)}{\tan C} = -\dfrac{\tan A}{\tan C}$, the negative sign indicating an external division of a line. Also,

$$\frac{AH}{HD} = 1 - \frac{AD}{HD} = 1 - \tan B\ \tan C\ = -\frac{\tan B\ + \tan C}{\tan A}.$$

When one angle, say ∡A, approaches 90° , we have $\dfrac{AH}{HD} = (\tan B\ + \tan C\,) \cot\ A\ = 0$. That is, A and H coincide, with AH = 0. As tan 90^0 is not defined, we work with its reciprocal, cot A.

Circumcentre

We now turn our attention to the circumcentre. In Figure 4, O is the circumcentre of acute angled Δ ABC. AO produced meets BC at K. BO produced meets CA at L, while CO produced meets AB at M.

Now, BK : KC = area Δ BOK : area Δ COK = $\frac{1}{2}$ r(OK) sin ∡BOK : $\frac{1}{2}$ r(OK) sin ∡COK = sin ∡BOK : sin ∡COK = sin ∡AOB : sin ∡AOC = sin 2C : sin 2B.

(Angle subtended at the centre by a chord is twice that subtended at a point on the circumference.)