ClassRoom

Euler's Inequality for the Circumradius and Inradius of a Triangle

SHAILESH SHIRALI

For any arbitrary triangle ABC, let R denote its circumradius and r its inradius (Figure 1). It was the Swiss-German mathematician Leonhard Euler who first observed that regardless of the shape of the triangle, the following inequality is invariably true:

$$R \geq 2r,$$

equality precisely when the triangle is equilateral.

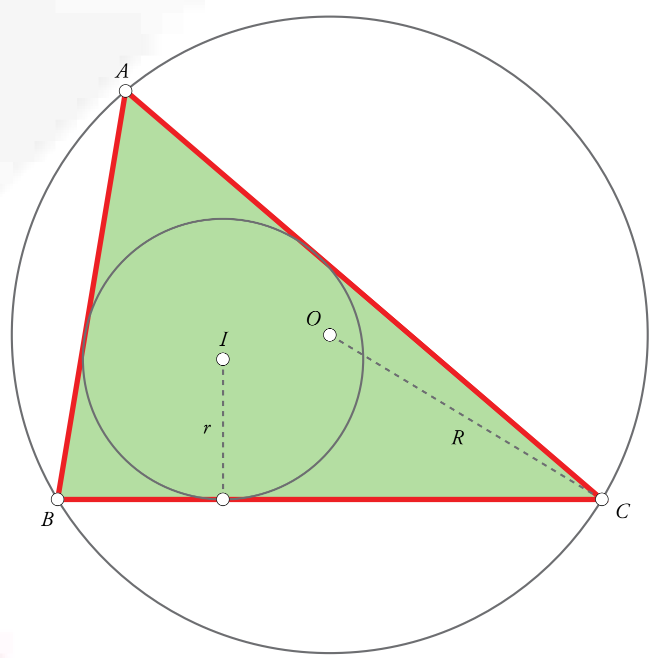


Figure 1.

Keywords: Euler, inequality, circumcircle, circumradius, incircle, inradius

The statement being so simple, one naturally longs for an equally simple proof of the result. Unfortunately, this is not readily forthcoming. (There are elegant and short proofs, but not simple proofs!) In this article, we present two very different proofs.

Notation. We use standard symbols for the various elements of the triangle: A, B, C for the three angles; a, b, c for the three sides (named appropriately); s for the semi-perimeter; R for the circumradius; r for the inradius; and Δ for the area of the triangle.

First proof. We first present a proof which has a strong component of algebra. The following formulas are all very well-known:

$$\begin{aligned}
2R &= \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},\\
\Delta &= \frac{1}{2}\,bc\sin A = \frac{1}{2}\,ca\sin B = \frac{1}{2}\,ab\sin C,\\
\Delta &= rs,\\
\Delta &= \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$

Combining the results in the first two lines, we obtain the additional result

$$\Delta = \frac{abc}{4R}.$$

In addition, we shall need the most basic result in the theory of inequalities, namely, the arithmetic mean-geometric mean inequality. This is the statement that if x, y are any two non-negative real numbers, then

$$\frac{x+y}{2} \geq \sqrt{xy},$$

with equality precisely when $x = y$. This gives rise to the following nice result. Let x, y, z be any three non-negative real numbers. Then we have:

$$\begin{aligned}
x+y &\geq 2\sqrt{xy},\\
y+z &\geq 2\sqrt{yz},\\
z+x &\geq 2\sqrt{zx}.
\end{aligned}$$

Hence by multiplication of the respective sides we obtain:

$$\boxed{(x+y)(y+z)(z+x) \geq 8xyz.} \tag{1}$$

Moreover, equality will hold in (1) precisely when $x = y = z$. Note that this is an interesting result in its own right.

Next we obtain the lengths of some segments associated with the incircle of a triangle (see Figure 2). Let D, E, F denote the points of contact of the incircle with the sides of the triangle, and let x, y, z denote the lengths of the segments as indicated.

It is easy to obtain x, y, z in terms of a, b, c. We have:

$$\begin{aligned}
y+z &= a,\\
z+x &= b,\\
x+y &= c.
\end{aligned}$$

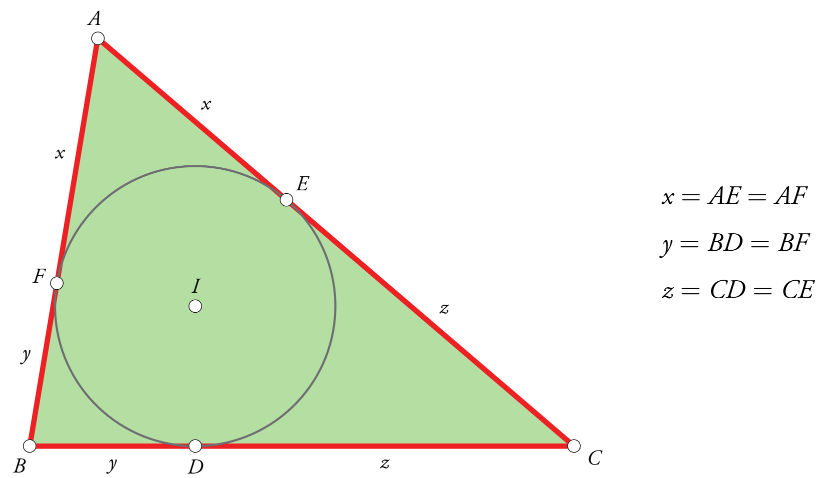


Figure 2.

By addition we get $2(x+y+z) = a+b+c = 2s$, hence $x+y+z = s$. Therefore:

$$x = s-a, \quad y = s-b, \quad z = s-c. \tag{2}$$

We now use these expressions for x, y, z in the inequality (1) (note that x, y, z are all positive; this follows from the triangle inequality, $b+c > a$, which yields $s-a > 0$; similarly for $s-b$ and $s-c$). We obtain:

$$abc \geq 8(s-a)(s-b)(s-c). \tag{3}$$

This too is an interesting result in its own right; it holds for any triangle. Moreover, equality holds precisely when $a = b = c$, i.e., when the triangle is equilateral.

We are now in a position to obtain Euler's inequality. From the formulas stated earlier for the area of a triangle, we have:

$$\Delta = \frac{abc}{4R},$$
$$\therefore \quad abc = 4R\Delta,$$

and:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$
$$\therefore \quad (s-a)(s-b)(s-c) = \frac{\Delta^2}{s}.$$

Therefore we have:

$$4R\Delta \geq \frac{8\Delta^2}{s},$$
$$\therefore \quad R \geq \frac{2\Delta}{s}.$$

Since $\Delta = rs$, the last line yields the desired result:

$$R \geq 2r, \tag{4}$$

with equality precisely when the triangle is equilateral.

Second proof. In contrast to the above, we now present a proof which is highly geometric. This is Euler's original proof (1765). He obtains the inequality as an easy consequence of an important geometric result.

Theorem (Euler). *The distance d between the circumcentre and the incentre of a triangle is related to its circumradius R and its inradius r by the following relation:*

$$d^2 = R(R-2r).$$

Remark. Before proceeding with the proof of the theorem, we note that the result instantly provides a proof of Euler's inequality; for, we must have $d^2 \geq 0$, and this yields $R \geq 2r$.

Proof of Euler's theorem. In Figure 3, let AI extended meet the circumcircle at L; let LO extended meet the circumcircle at M; let segment IO extended in both directions meet the circumcircle at P and Q; and finally, let F be the foot of the perpendicular from I to AB.

Consider $\triangle AFI$ and $\triangle MBL$. They are similar to each other, for $\measuredangle AFI$ and $\measuredangle MBL$ are right angles, and $\measuredangle FAI = \measuredangle BML$ ("angles in the same segment"). Hence:

$$\frac{FI}{BL} = \frac{AI}{ML}, \quad \text{i.e.,} \quad \frac{r}{BL} = \frac{AI}{2R}, \tag{5}$$

which yields $2Rr = AI \cdot BL$. Next, we claim that $BL = IL$, i.e., that $\triangle LBI$ is isosceles. This follows from a simple computation of angles. For we have,

$$\measuredangle LBI = \measuredangle LBC + \measuredangle IBC = \measuredangle LAC + \measuredangle IBC = \frac{\measuredangle A}{2} + \frac{\measuredangle B}{2},$$

and

$$\measuredangle LIB = \measuredangle LAB + \measuredangle ABI = \frac{\measuredangle A}{2} + \frac{\measuredangle B}{2}.$$

It follows that $BL = IL$ and so:

$$\boxed{2Rr = AI \cdot IL.} \tag{6}$$

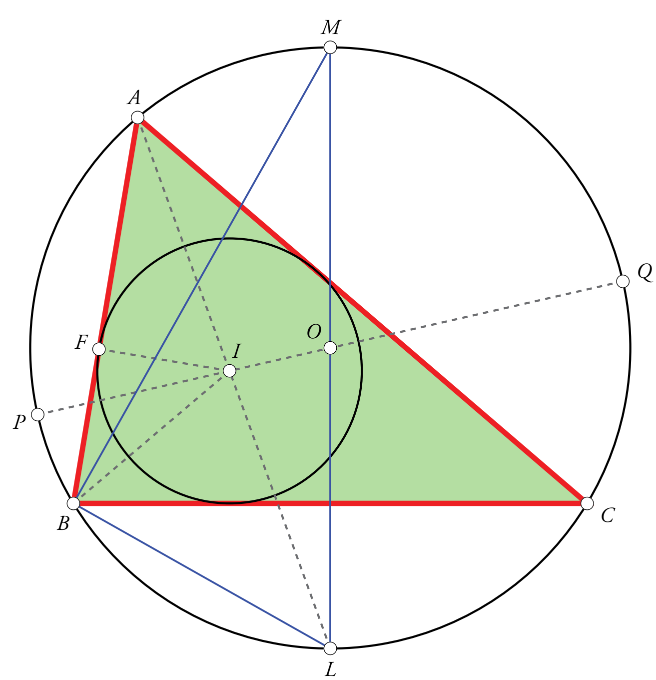


Figure 3.

Finally, from the intersecting chords theorem it follows that

$$AI \cdot IL = PI \cdot IQ.$$

Since $PI = OP - OI = R - d$ and $IQ = QO + OI = R + d$, we obtain:

$$2Rr = (R - d)(R + d) = R^2 - d^2,$$

i.e.,

$$\boxed{d^2 = R^2 - 2Rr = R(R - 2r).} \tag{7}$$

Euler's inequality now follows.

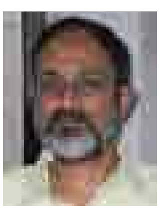
For more proofs, the three references listed below may be consulted.

References

1. Wikipedia, "Euler's theorem in geometry" (from Wikipedia, the free encyclopedia), https://en.wikipedia.org/wiki/Euler's_theorem_in_geometry

2. Wikipedia, "Incircle and excircles of a triangle" (from Wikipedia, the free encyclopedia), https://en.wikipedia.org/wiki/Incircle_and_excircles_of_a_triangle

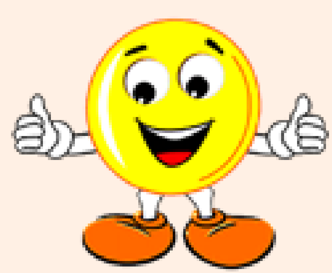
3. Samer Seraj, "A Short Proof of Euler's Inequality", *Resonance – Journal of Science Education*, Volume 20, Issue 1, January 2015 pp 75-75, https://www.ias.ac.in/article/fulltext/reso/020/01/0075-0075



SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Math Jokes and Puns

1. Why was the fraction apprehensive about marrying the decimal?
   Because he would have to convert...
2. Why do plants hate math?
   Because it gives them square roots...
3. Why did the student get upset when his teacher called him average?
   Well, it was a pretty mean thing to say!
4. Why was the math book depressed?
   Poor thing, it had too many problems.
5. Why is the obtuse triangle always so frustrated?
   Because it is never right.
6. Why can you never trust a math teacher holding graphing paper?
   He must be plotting something.



Contributed by Harin Hattangady, Azim Premji University, M.A. Education, batch of 2011-13