Problem Corner

Problems for the Senior School

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Problem IX-1-S.1

The midpoints of two sides of a triangle are marked. How can the midpoint of the third side be found using only a pencil and a straightedge?

Problem IX-1-S.2

Is it possible to cut several circles out of a square of side 10 cm, so that the sum of the diameters of the circles is 5 metres or more?

Problem IX-1-S.3

Suppose in a given collection of 2020 integers, the sum of every 100 of them is positive. Is it true that the sum of all the 2020 integers is necessarily positive?

Problem IX-1-S.4

Suppose integers a, b and c are such that $ax^2 + bx + c$ is divisible by 5 for any integer x. Prove that each of a, b and c is divisible by 5.

Problem IX-1-S.5

The altitude dropped from A onto BC in triangle ABC is not shorter than BC, and the altitude dropped onto AC from B is not shorter than AC. Find the angles of triangle ABC.

Keywords: Triangle, midpoint, circle, integer, altitude, divisible, octagon, circumcircle

Solutions of Problems in Issue-VIII-3 (November 2019)

Solution to problem VIII-3-S.1

From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Determine the length of a side of the octagon.

Let t be the length of a side of the regular octagon. The four triangular pieces removed are four congruent right-angled isosceles triangles with hypotenuse t. It follows that

$$t + 2\left(\frac{t}{\sqrt{2}}\right) = 5$$

whence $t = 5(\sqrt{2} - 1)$.

Solution to problem VIII-3-S.2

Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1, respectively, and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is $40°$, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees? (Figure 1.)

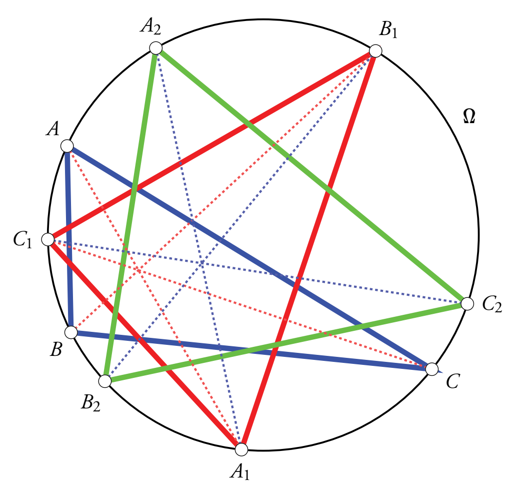


Figure 1.

Simple angle chasing yields

$$\measuredangle A_1 = 90° - \frac{1}{2}\measuredangle A, \quad \measuredangle B_1 = 90° - \frac{1}{2}\measuredangle B, \quad \measuredangle C_1 = 90° - \frac{1}{2}\measuredangle C,$$
$$\therefore\ \measuredangle A_2 = 90° - \frac{1}{2}\measuredangle A_1 = 45° + \frac{1}{4}\measuredangle A,$$

and in the same way,

$$\measuredangle B_2 = 45° + \frac{1}{4}\measuredangle B, \quad \measuredangle C_2 = 45° + \frac{1}{4}\measuredangle C.$$

If $\measuredangle A$ is the smallest angle of $\triangle ABC$ then $\measuredangle A_2$ is the smallest angle of $\triangle A_2B_2C_2$. Therefore, $\measuredangle A_2 = (45 + 40/4)° = 55°$.

Solution to problem VIII-3-S.3

The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of ABC. Determine the angles of the triangle ABC. (Figure 2.)

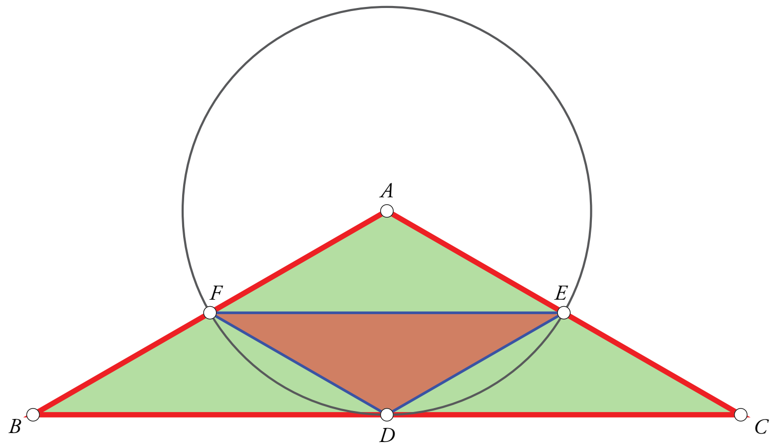


Figure 2.

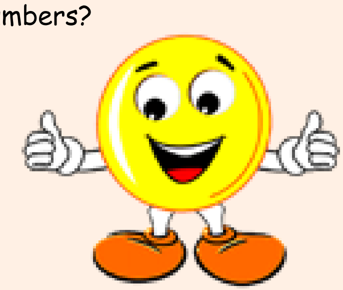
In $\triangle ABC$, let $AB = AC$. Suppose D is the midpoint of BC, E is the midpoint of AC and F is the midpoint of AB. Then DEF is similar to ABC with $\measuredangle D = \measuredangle A$. Also, EF is parallel to BC. The perpendicular bisector l of BC passes through A and it is also the perpendicular bisector of EF. Therefore the circumcentre of $\triangle DEF$ lies on l. As it lies on the circumcircle of ABC, it is outside $\triangle DEF$. Thus $\measuredangle D$ is obtuse which implies that the circumcentre lies on the opposite side of EF as D. It is that point of intersection of l and the circumcircle of $\triangle ABC$ which is on the opposite side of EF as D. Therefore it must be A. Hence:

$$360^\circ - \measuredangle A = 2\measuredangle A,$$

so $\measuredangle A = 120^\circ$. Thus the angles of ABC are $120^\circ, 30^\circ, 30^\circ$.

Math Jokes and Puns

1. Why was the equal sign so humble?
   Because she knew that she wasn't greater than or less than anyone else.
2. What do you call the number 7 and the number 3 when they go on a date?
   The odd couple (but 7 is in her prime).
3. I'll do algebra, I'll do trig, I'll even do statistics…
   … But graphing is where I draw the line!
4. Why should you never talk to Pi?
   Because he goes on and on and on forever…
5. Why are parallel lines so tragic?
   Because they never get to meet…
6. What is the best way to flirt with a math teacher?
   Use acute angle.
7. Did you hear about the mathematician who is afraid of negative numbers?
   He stops at nothing to avoid them.
8. How do you stay warm in any room?
   Just huddle in the corner, where it's always 90 degrees.
9. Why is six afraid of seven?
   Because seven eight ("ate") nine!
10. Why DID seven eat nine?
   Because you're supposed to eat 3 squared meals a day!



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