Problem Corner

Adventures in Problem Solving

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In this edition of 'Adventures' we study three problems. As always, we pose the problems first and present the solutions later. The different approaches used to solve the problems should be studied with care by the student.

Miscellaneous problems

Problem 1. Determine the dimensions of all integer-sided cuboids whose surface area is 100 square units.

Problem 2. Rectangle $ABCD$ has sides $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\measuredangle BPC = 90°$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?

Problem 3. Let O be the midpoint of the base BC of an isosceles triangle ABC. A circle is drawn with centre O and tangent to the equal sides AB and AC. Let P be a point on AB and Q a point on AC such that PQ is tangent to this circle. Prove that $BP \cdot CQ = \frac{1}{4}BC^2$. Discuss the converse of this result. (Australia, 1981)

Solutions to the problems

Solution to problem 1

Let the dimensions of the cuboid be a, b, c where a, b, c are positive integers, labelled so that $a \leq b \leq c$. Then we have $2ab + 2bc + 2ca = 100$, and therefore

$$ab + bc + ca = 50.$$

Keywords: Cuboids, surface area, rectangles, incircles, isosceles triangles

Since $a = \min(a, b, c)$, it must be that $ab + bc + ca \geq 3a^2$. Therefore we get $3a^2 \leq 50$. This implies that $a \leq 4$. So a must take one of the values $1, 2, 3, 4$. We consider each possibility in turn. In each case, the clinching argument will be through a suitable factorisation.

a = 1: The equation in this case reduces to $bc + b + c = 50$. We employ a clever but familiar trick now, by adding 1 to both sides. The left side then becomes $bc + b + c + 1$, and this expression factorises as $(b + 1)(c + 1)$. Hence it must be that

$$(b + 1)(c + 1) = 51.$$

Therefore $b + 1, c + 1$ are a pair of positive integers whose product is 51, and $1 \leq b \leq c$. The factorisation 1×51 does not work, as we must have $b \geq 1$, i.e., $b + 1 \geq 2$. The only other factorisation available is 3×17, and this yields $b + 1 = 3$, $c + 1 = 17$, so the triple in this case is $(a, b, c) = (1, 2, 16)$.

a = 2: The equation in this case reduces to $bc + 2b + 2c = 50$, or

$$(b + 2)(c + 2) = 54.$$

Now we must look for factorisations of 54, the smaller factor being at least 4 (since we must have $b \geq 2$). The only possibility available is $6 \times 9 = 54$, which gives $b + 2 = 6$ and $c + 2 = 9$. Hence the triple in this case is $(a, b, c) = (2, 4, 7)$.

a = 3 or 4: The remaining two cases may be analysed in the same way but they do not yield any fresh solutions. (Details left to the reader.)

So there are just two such cuboids, their dimensions being $(1, 2, 16)$ and $(2, 4, 7)$. It should be clear that this approach will work for any specified surface area.

Solution to problem 2

Please see Figure 1. Observe that

$$r_1 = \frac{AP + AB - BP}{2}, \quad r_2 = \frac{BP + PC - BC}{2}, \quad r_3 = \frac{DP + CD - PC}{2}.$$

Note that these relations hold because $\triangle BAP$, $\triangle BPC$ and $\triangle CPD$ are all right-angled. (The general statement is the following: if $\triangle ABC$ is right-angled, with the right angle at A, then the inradius r is given in terms of the sides a, b, c by the relation $2r = b + c - a$.) Adding these we obtain

$$r_1 + r_2 + r_3 = \frac{AD + AB + CD - BC}{2} = AB = 8.$$

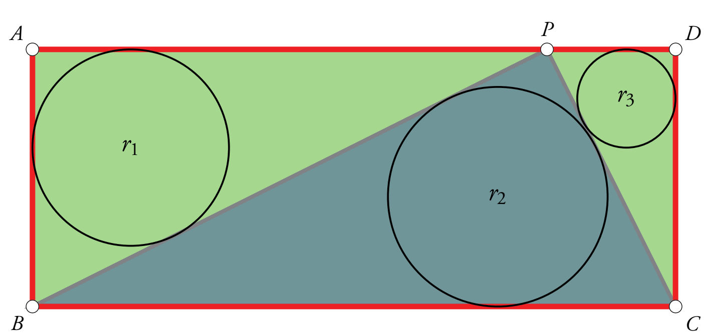

Figure 1.

Solution to problem 3

Please see Figure 2. Let L, M be the points of tangency of AB, AC with the semicircle, and let PQ touch the semicircle at D. Join OP and OQ, as shown.

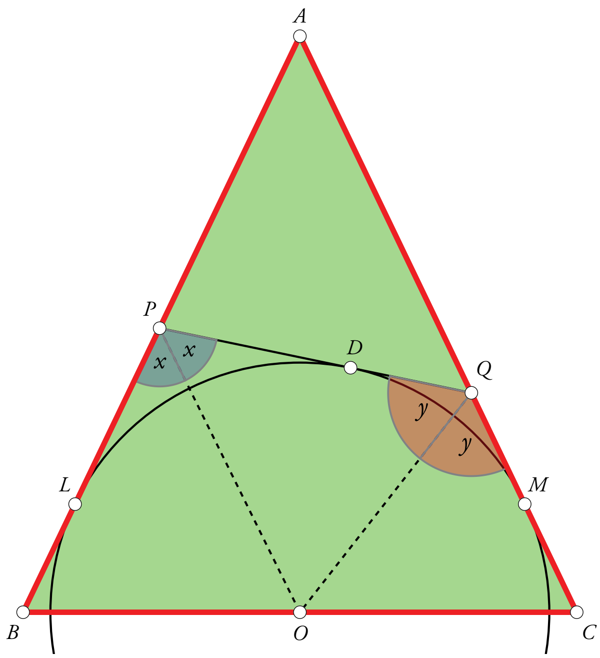

Figure 2.

Let $\measuredangle ABC = \theta = \measuredangle ACB$, $\measuredangle BPQ = 2x$, $\measuredangle CQP = 2y$. Consider quadrilateral $BPQC$. Its angles add up to 360°, therefore

$$2x + 2y + \theta + \theta = 360^\circ, \quad \therefore\ x + y + \theta = 180^\circ.$$

We now have:

$$\measuredangle BPO = x = \measuredangle OPQ, \quad \measuredangle OQP = y = \measuredangle OQC,$$
$$\therefore\ \measuredangle POQ = 180^\circ - x - y = \theta.$$

It follows that

$$\measuredangle BOP = y, \quad \measuredangle COQ = x,$$
$$\therefore\ \triangle PBO \sim \triangle POQ \sim \triangle OCQ,$$
$$\therefore\ \frac{PB}{BO} = \frac{OC}{CQ},$$
$$\therefore\ BP \cdot CQ = BO \cdot OC = \frac{1}{4} BC^2.$$

Note that we have not answered the question about the converse. We leave this part for the reader to explore.

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