Secondary

Student Corner

Pell's Equation

RAHIL MIRAJ

Introduction. In number theory, a Diophantine equation is one for which integer solutions are sought (or, sometimes, solutions in rational numbers). Typically, the number of variables is greater than the number of equations, allowing for the possibility of infinitely many solutions. The task of finding these solutions can sometimes be very challenging.

A very famous quadratic Diophantine equation is Pell's equation, which has the form

$$x^2 - dy^2 = 1. \tag{1}$$

Here, d is a given natural number which is not a perfect square, and x, y are required to be non-negative integers. We can see that the solution $x = 1, y = 0$ works for every value of d, and hence is not of much interest; we call it a 'trivial solution.' So we shall look only for nontrivial solutions of the equation.

This work is an extension of a previous work [2], where I had suggested some ways for finding the initial solution of Pell's equation.

If one nontrivial integral solution (x_1, y_1) is found, then one can find infinitely many nontrivial solutions from the following formula [1]:

$$x_n = \frac{\left(x_1 + y_1\sqrt{d}\right)^n + \left(x_1 - y_1\sqrt{d}\right)^n}{2}, \qquad y_n = \frac{\left(x_1 + y_1\sqrt{d}\right)^n - \left(x_1 - y_1\sqrt{d}\right)^n}{2\sqrt{d}}. \tag{2}$$

This formula is obtained by equating (respectively) the rational and the irrational parts on the two sides of the following (after expanding the expression on the right side):

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n. \tag{3}$$

Keywords: Diophantine equation, trivial solution, nontrivial solution, Python programming language, infinite loop

Illustrating the use of formula (3)**.**

d = 3: Here the equation is $x^2 - 3y^2 = 1$. An obvious initial solution is $x = 2, y = 1$. To generate more solutions, we consider the integral powers of $2 + \sqrt{3}$:

$$\left(2+\sqrt{3}\right)^2 = 7 + 4\sqrt{3}, \text{ therefore } x = 7,\ y = 4 \text{ is a solution;}$$

$$\left(2+\sqrt{3}\right)^3 = 26 + 15\sqrt{3}, \text{ therefore } x = 26,\ y = 15 \text{ is a solution; } \ldots.$$

d = 5: Here the equation is $x^2 - 5y^2 = 1$. An initial solution is $x = 9, y = 4$. To generate more solutions, we consider the integral powers of $9 + 4\sqrt{5}$:

$$\left(9+4\sqrt{5}\right)^2 = 161 + 72\sqrt{5}, \text{ therefore } x = 161,\ y = 72 \text{ is a solution;}$$

$$\left(9+4\sqrt{5}\right)^3 = 2889 + 1292\sqrt{5}, \text{ therefore } x = 2889,\ y = 1292 \text{ is a solution; } \ldots.$$

But it can be quite difficult to find an initial solution, particularly if d is large. Some kind of heuristic is needed.

A straightforward restatement of Pell's equation is the following. We need to look for integers $t > 1$ such that the quantity y given by

$$y = \sqrt{\frac{t^2 - 1}{d}} \tag{4}$$

is an integer. That is, we must have $d \mid t^2 - 1$, and the quotient $\frac{t^2-1}{d}$ must be a perfect square. Then, obviously, (x, y) is a solution to Pell's equation. The challenge is to find a feasible value of t in some easy manner. My explorations have helped me find some general heuristics.

- If d is of the form $k^2 + 1$, choose $t = 2k^2 + 1$; then $y = 2k$.
- If d is of the form $k^2 - 1$, choose $t = k$; then $y = 1$.
- If d is of the form $k^2 + 2$, choose $t = k^2 + 2$; then $y = k$.
- If d is of the form $k^2 - 2$, choose $t = k^2 - 1$; then $y = k$.

Here k is any positive integer. This heuristic works provided $k^2 - 2 \le d \le k^2 + 2$ for some integer $k \ge 2$. But it is difficult to choose the values of t for other forms of d.

To overcome this, I wrote a computer program, using the Python programming language. To see the code, please refer to the appendix.

The program starts with $t = 2$ and checks whether it works; if it does not, it increases the value of t by 1 and checks whether the new value of t works; and it continues in this manner indefinitely. The program thus runs in an infinite loop, with no upper limit; it keeps running till it finds a solution. So the program takes time according to the choice of d as well as the configuration of the computer.

Examples. We find these solutions of Pell equations using the computer program.

- For $d = 2$, we choose $t = 4$ and get $(x, y) = (3, 2)$.
- For $d = 3$, we choose $t = 3$ and get $(x, y) = (2, 1)$.
- For $d = 5$, we choose $t = 10$ and get $(x, y) = (9, 4)$.
- For $d = 33$, we choose $t = 24$ and get $(x, y) = (23, 4)$.
- For $d = 2019$, we choose $t = 675$ and get $(x, y) = (674, 15)$.

Comment. For some values of d, even the smallest solution has extremely large values of x, y, so it is not practical to look for a solution in this manner. For example:

- For $d = 85$, we get $(x, y) = (285769, 30996)$.
- For $d = 1000$, we get $(x, y) = (39480499, 1248483)$.
- For $d = 1729$, we get $(x, y) = (544796401, 13101974)$. (Here, 1729 is the famous 'Ramanujan number.')

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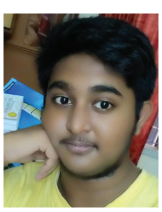
Appendix: Python code

```
import math
d=int(input("d="))
t=2
while True:
    m=math.floor(math.sqrt(d))
    p=t-1
    q=(p*(t+1))/d
    x=t
    y=(math.sqrt(q))
    h=(math.floor(y))
    if m*m==d:
        print (d, "is␣the␣square␣of␣an␣integer.")
        break
    elif h*h==q:
        print("Here␣we␣have␣to␣choose␣t=",t)
        print("x=",x)
        print("y=",y)
        break
    else:
        t+=1
```

Comment. The 't' in the above code and the above analysis differs by 1 from the 't' in reference [2]. So 't = 2' in the above presentation corresponds to 't = 3' in the original reference [2]. Similarly, the expression $(t^2 - 1)/d$ above corresponds to the earlier $(t^2 - 2t)/d$, and so on. This has been done to simplify the presentation.

References

1. Titu Andreescu, Dorin Andrica, Ion Cucurezeanu, An Introduction to Diophantine Equation, Birkhäuser,(2010).
2. Rahil Miraj, International Journal of Mathematics Trends and Technology, 59, 45,(2018).



RAHIL MIRAJ is a student of Class-XI in Sarada Vidyapith (H.S), Sonarpur, Kolkata-700150. He is highly interested in Pure Mathematics and writes articles on Mathematics in local magazines in Bengali. He has published seven papers in various journals including News Bulletin of Calcutta Mathematical Society and At Right Angles. He has also presented papers in two international and three local conferences. He is also interested in solving cubes, playing computer games, writing Python code, and doing various experiments on Physics. He may be contacted at rahilmiraj@gmail.com.