Secondary

Student Corner

Study of an Inequality Problem from an Olympiad – Part 1

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During one of the meetings of a problem solving group in our school, we were given a problem from the British Mathematics Olympiad Round 2, 2004. We present three solutions to the problem. The first one was obtained during the session itself, and the other two were developed through discussions. In a follow-up article, we shall explore another BMO problem which we solved using a similar idea.

Problem 1. Given real numbers a, b, c with $a + b + c = 0$, prove that

$$a^3 + b^3 + c^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 > 0.$$

Solution 1. This is an 'ad hoc' solution. Let $a + b + c = 0$; then $c = -(b + a)$. We assume (without any loss in generality) that $a \geq b \geq c$. It follows that $a \geq 0$. If $a = 0$, then $b = 0 = c$ as well, in which case we have $a^3 + b^3 + c^3 = 0 = a^5 + b^5 + c^5$; so this case need not be considered. Hence we may as well suppose that $a > 0$. We now have,

$$\begin{aligned} a^3 + b^3 + c^3 &= a^3 + b^3 - (a + b)^3 \\ &= -3ab(a + b). \end{aligned} \tag{1}$$

Next,

$$\begin{aligned} a^5 + b^5 + c^5 &= a^5 + b^5 - (a + b)^5 \\ &= -5a^4b - 10a^3b^2 - 10a^2b^3 - 5ab^4 \\ &= -5ab(a + b)\left(a^2 + b^2 + ab\right). \end{aligned} \tag{2}$$

We claim that $a^2 + ab + b^2 > 0$. For:

- If $b = 0$, then $a^2 + ab + b^2 = a^2 > 0$, since $a > 0$.
- If $b > 0$, then $ab > 0$, so $a^2 + b^2 + ab > 0$.
- If $b < 0$, then $c < 0$ as well, since $c \leq b$ (by supposition). Since $b = -a - c > -a$, it follows that $|b| < |a|$. This implies $|ab| < |a^2|$, therefore, $a^2 + ab > 0$.

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Thus $a^2+ab+b^2>0$ as claimed. Now, from (1) and (2), we observe that

$$\begin{aligned} a^5+b^5+c^5>0 &\iff 5ab(a+b)<0 \iff ab(a+b)<0 \\ &\iff -3ab(a+b)>0 \iff a^3+b^3+c^3>0. \end{aligned}$$

Solution 2. Here we follow the method given in [2] (*Higher Algebra*, Hall and Knight). Consider the following identity

$$(1+ax)(1+bx)(1+cx)=1+qx^2+rx^3,$$

where $q=ab+bc+ca$, $r=abc$ (there is no x-term since $a+b+c=0$). Taking logarithms of both sides, we get:

$$\log(1+ax)+\log(1+bx)+\log(1+cx)=\log\left(1+qx^2+rx^3\right).$$

We now use the logarithmic series expansion (valid for all $|x|<1$),

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

for each term. (Note. For the series expansion to work, we also need to have $|ax|<1$, $|bx|<1$, $\ldots$, $|rx^3|<1$. This does not present a problem, because a, b, c, q, r are fixed quantities, and we can always take x to be small enough so that these inequalities are satisfied.) Equating the coefficients of x^n on both sides, we find that

$$(-1)^{n-1}\left(a^n+b^n+c^n\right)$$

is equal to n times the coefficient of x^n in

$$\left(qx^2+rx^3\right)-\frac{1}{2}\left(qx^2+rx^3\right)^2+\frac{1}{3}\left(qx^2+rx^3\right)^3-\cdots.$$

Now put $n=2,3,4,5$ in the above; we get the following:

$$\begin{aligned} a^2+b^2+c^2&=-2q,\\ a^3+b^3+c^3&=3r,\\ a^4+b^4+c^4&=2q^2,\\ a^5+b^5+c^5&=-5qr. \end{aligned}$$

We observe that

$$3\left(a^2+b^2+c^2\right)\left(a^5+b^5+c^5\right)=3(-2q)(-5qr)=30q^2r,$$

and also

$$5\left(a^3+b^3+c^3\right)\left(a^4+b^4+c^4\right)=30q^2r.$$

Thus we have established the following beautiful relation: if $a+b+c=0$, then

$$5\left(a^3+b^3+c^3\right)\left(a^4+b^4+c^4\right)=3\left(a^2+b^2+c^2\right)\left(a^5+b^5+c^5\right).$$

Since $a^4+b^4+c^4$ and $a^2+b^2+c^2$ are never negative, and are 0 precisely when a, b, c are all 0, the assertion in the problem follows.

Solution 3. The strategy is to use the symmetric functions of the roots of a polynomial. First we denote for all $k\geq 0$,

$$S_k=a^k+b^k+c^k.$$

Thus we have $S_0=3$, $S_1=0$, $S_2=a^2+b^2+c^2$, etc., and a, b, c are the roots of the cubic

$$f(x)=(x-a)(x-b)(x-c)=x^3-S_1x^2+(ab+bc+ca)x-abc,$$

i.e., the cubic

$$f(x)=x^3+(ab+bc+ca)x-abc.$$

Therefore, we have

$$a^3 + (ab + bc + ca)a - abc = 0.$$

Multiplying by a^k, we get,

$$a^{k+3} + (ab + bc + ca)a^{k+1} - (abc)a^k = 0, \tag{3}$$

for $k \geq 0$. Similarly, we have

$$b^{k+3} + (ab + bc + ca)b^{k+1} - (abc)b^k = 0. \tag{4}$$

and

$$c^{k+3} + (ab + bc + ca)c^{k+1} - (abc)c^k = 0. \tag{5}$$

By adding these three relations, we get, for $k \geq 0$:

$$S_{k+3} = -(ab + bc + ca)S_{k+1} + (abc)S_k. \tag{6}$$

Put $k = 2$ in (6):

$$S_5 = -(ab + bc + ca)S_3 + (abc)S_2. \tag{7}$$

Now observe that

$$2(ab + bc + ca) = (a + b + c)^2 - \left(a^2 + b^2 + c^2\right) = 0 - S_2 = -S_2,$$

which simplifies to

$$ab + bc + ca = -\frac{1}{2}S_2.$$

and from $a^3 + b^3 + c^3 = 3abc$, we get

$$abc = \frac{1}{3}S_3.$$

Now, using these in equation (7), we have

$$S_5 = \frac{1}{2}S_2S_3 + \frac{1}{3}S_2S_3 = \frac{5}{6}S_2S_3 \tag{8}$$

Now note that $S_2 = 0$ implies $a = b = c = 0$. In that case, we get $S_3 = S_5 = 0$. If a, b, c are not all 0, then S_2 is a positive number, so S_5 is a positive number times S_3. It follows that

$$S_5 > 0 \iff S_3 > 0,$$

as required.

Comment from the editor. Observe that the strategy followed in the third solution is the same as that followed in the "extreme algebra" problem explored elsewhere in this issue.

Exercises for the reader

(1) Given real numbers a, b, c with $a + b + c = 0$, prove that

$$a^7 + b^7 + c^7 > 0 \iff a^5 + b^5 + c^5 > 0.$$

(2) Given real numbers a, b, c, d with $a + b + c + d = 0$, prove that

$$a^3 + b^3 + c^3 + d^3 > 0 \iff a^5 + b^5 + c^5 + d^5 > 0.$$

(3) Given real numbers a, b, c with $a + b + c = 0$, find all $k \in N$, such that

$$a^k + b^k + c^k > 0 \iff a^3 + b^3 + c^3 > 0.$$

(4) Given real numbers a, b, c with $a + b + c = 0$, find all $k, m \in N$, such that

$$a^k + b^k + c^k > 0 \iff a^m + b^m + c^m > 0.$$

References

1. UK Mathematics Trust, https://www.ukmt.org.uk/
2. H. S. Hall and S. R. Knight, *Higher Algebra*, Macmillan, 1964.

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