Area of Rectangle = Length × Breadth: Conversations with a 9 year old

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Children are taught many kinds of measures in the primary grades, like length, weight, volume, money, time and finally area and perimeter (which is nothing but measurement of length). While for teaching length, weight and volume, efforts are made to give students an exposure to informal/ non-standard units of measurement (and thus some implicit understanding of what we may use to measure something), the other measurement ideas start with the standard units, conversions and formulae. Even when non-standard units are introduced, rarely is emphasis laid on explicating the principles which are routinely followed in measuring – like choosing a unit and iterating it repeatedly without leaving any gaps or overlaps. It may be assumed that since children already know about informal/ non-standard units for some aspects/ dimensions of measurement, they will now be ready for dealing with standard units for all kinds of measurement, including time and area. Also, measurement of time and area are more difficult ideas to teach and learn. "Time" is often reduced to reading and understanding the clock, without a focus on what is meant by time and measuring time. Similarly, area and perimeter are more about formulae than about understanding what area and perimeter are, how they are different or connected. The following piece describes my attempt to engage my 9-year old daughter (M) in thinking about area and perimeter. My goals are to illustrate some challenges we face while attempting to teach children these ideas, suggest some instructional ideas as well as give a peek into a child's thinking.

Keywords: Measurement, units, standard, non-standard, area, perimeter, formulas, mathematical conversations

The context

It all started with a visit of my family to a newly constructed house and a conversation about the carpet area of the house. As the adults discussed the carpet area, M asked us what square-feet meant? She was aware of quite a few measuring units, including units for length, weight, volume. She had recently studied perimeter of rectilinear and circular shapes also (whose unit was the same as of length) but could not guess what such a unit as "square-feet" could mean.

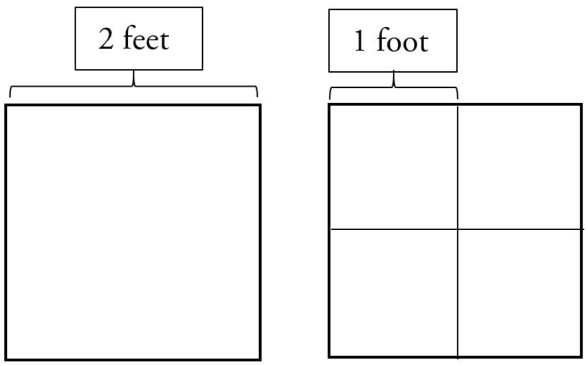

Figure 1: Schematic figure of tiles and partitioning

I quickly explained that each of the tiles on the floor was 2 feet long on each side and the space occupied by the entire square is 4 square feet. Further, I went on to say that if we were to make 4 equal parts of the square tile, so that each part still remained a square (which meant that the equal parts have to be created by one horizontal line and one vertical line as shown in Figure 1), then the area of the smaller square would be 1 square foot. In effect, I defined "square-foot" as the area of a square whose sides were 1 foot each. I told her that if she were to count the total number of squares of dimension 1 foot each on the floor of the house, she would find the carpet area of the house. We did not have the time to do that, so we moved on.

A couple of weeks later, she was introduced to area in school. She understood that area is the space inside the boundary of the shape. Rectangle and square were the only two shapes they were introduced to in this context. Also, they wrote the formulae for finding the area of a rectangle and a square. They were kept separate and she did not believe that there is any relation between rectangle and square; one had opposite sides equal and parallel while the other had all sides equal. On different occasions, I have tried to push her to think why square is also a rectangle and why the same formula can be used in the contexts of perimeter and area. I must admit that I have not been very successful in helping her see the square as a special case of rectangle, where all properties of rectangle remain, with an additional property that all sides are equal. She in fact counter-argues that if they were the same, then why were there two different formulae in both the contexts. Though the relation between square and rectangle and many other quadrilaterals will be explored in the years to come in the context of geometry, we start seeing the limitations of teaching concepts like area and perimeter using formulae as the basis.

Being a mathematics teacher, I got my act together and decided to teach M what area and perimeter of such rectilinear figures are. It was simple to deal with perimeter as she understood that perimeter is the "total boundary" which could be found by adding the lengths of the sides for a rectilinear figure; and in the case of a circular shape, one could use a thread around the boundary and measure the length of the thread using the ruler. Therefore, it was area that became my focus.

Introduction to measuring area

I started by revising how length, weight and volume are measured by using a variety of non-standard units, that she had tried at school or home. This helped us see the kinds of measure that we were measuring, the choice of unit and the principles underlying measurement. For example, when we measure length, we use any of the following non-standard units: digit, cubit, hand-span, foot-span, strips of paper repeatedly placed along the object till we have exhausted the expanse of the length. M remembered that repeated iteration of a strip of paper, without leaving any gaps, either completely fitted the object or sometimes needed to be folded in some ways (some fractional part) to quantify the measure (length in this case). We had done this task quite exhaustively and therefore she vividly remembered the piece of paper that she had used to measure different objects in her room – the bed, the study table, the chair, etc. She could connect this folding process to fractions also, so this was a good way to introduce her to the idea of scale. Sometimes these revelations lead to a spark and surprise in her eyes and I am thrilled with it as a teacher. This is not to say that these insights stay with her always.

This revision was useful in understanding that area is also a measure of the space inside the boundary of the shape and therefore we need to identify a unit for measuring it. What could be the unit was the next question. M did not seem to recollect the conversation we had had about the carpet area of the empty house a few days back. She started by saying that we could use strips of paper as a unit. Prima facie, agreeing to the suggestion, we discussed how the strip of paper would look. In her imagination the strip had the same length as of the rectangle which was iterated within a bigger rectangle (shown below, Fig 2). I asked her: what is the area of the strip? Guessing that my objection was to the size of the strip, she suggested that we can reduce the strip to half. The question of the area of this smaller strip bugged her as she realized no matter how small the strip was, we still needed to know the area of the strip. And she gave up briefly.

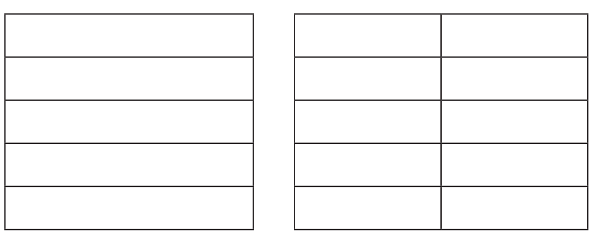

Figure 2

Leading to a choice of unit for measuring area

Before she could get out of this situation, I proposed what if instead of these strips, we use lines to fill the rectangle. She readily accepted this suggestion – she had found the thinnest possible strip of paper. I asked her how we would find the area then. She said "find the length of each of the line segments, and multiply by the number of lines!" This perhaps is closest to the formula of area of rectangle that we learn and in children's heads may be the rationale for why we multiply the length and the breadth of the rectangle to get the area. She of course realized that it was a daunting task, as there would be so many lines which will be required to fill the space inside a rectangle of a similar size. She was well aware of the way she had iterated the strip of paper to find the length of the object – there must not be any gaps.

This was an important point in this discussion and discovery, both for me and my daughter. The square unit that we use in the measurement of area is taken for granted. We need to identify and construct a unit for measurement. But before we could do that, I had to show the illegitimacy of using lines as the unit for measuring area. So, I asked her to tell me what a line is. She promptly gave me the definition that they had learnt in school – a line is straight which extends indefinitely in both directions. But the important Euclidean idea that line has no breadth but only one linear dimension is not emphasized in school so early. And therefore, the conflict between the concrete instantiation (physical manifestation) that we draw in notebooks and the ideal line is evident. The lines that we draw with our pencils/pens (however pointed) can fill the required space, a small breadth getting attributed to it as the tip of the graphite pencil scratches the surface of the paper but the "ideal" line will not. There was no possible way for me to proceed without explaining this aspect to her. She was amazed to realize this. I also pointed out in the same vein that the "ideal" point is dimensionless, no length, no depth, no height. But a mark on the paper, again with pen or pencil, gives the look that it has all of these.

So, we moved on from here to discuss if thread can be a unit of measure of area. Now she was quick to figure out the difficulty – that it indeed will fill the space but how are we to account for the thickness of the thread. I took her back to the strips of paper that she had initially suggested and we felt that was a better option, except for the fact that we needed some way to find the area of the strips as well. But in the meantime, we could place the strips inside the rectangle and state the area of the rectangle as (say in figure above) 10 rectangle units. (A bit weird!) we had some more conversations that day about what all shapes we could fill in the rectangle to measure the area, with the understanding that we will eventually need to find the area of each of these shapes also. Working with some materials like small split pulses, circular buttons, small squares and rectangles, she realized that iterations of some shapes (squares, rectangles) leave no space between them. Circular buttons left some space even when she used a smaller button to cover the gap between a few larger buttons. Split pulses could be arranged very close but still there were very small gaps. This was a good moment to say that the shapes must "tile", an idea that we explicitly explore in tessellation. Circles do not tile. Similarly, pentagons will not tile by themselves, unless we use some other shape together with it with specific dimensions. We can see lots of such examples on the footpaths. She quickly recalled a newly laid pavement with square tiles.

This conversation had moved us towards an understanding that not all shapes can be used as a unit of measuring area. It has to be rectangles, squares, and even right-angled triangles (if scalene, two of them will make a rectangle and if isosceles, two of them will make a square). By the end of that day, we had delimited the shapes that may serve as a unit of measuring area. I once again drew her attention to the square tile on the floor of the room and she measured to see how much of it was 1 foot. The tile was once again a square of side 2-feet, leading to the same situation of dividing the tile into four equal parts as discussed earlier. I now casually asked – what is the area of the small square with sides 1 foot? The response was baffling – "I know how to find the area with the formula but here how do I find?" With a little prodding and nudging she realized that the formula can be easily used here also, that what we learn in school is not just for mathematics notebooks but for the world outside as well. After she found the area of the square with unit length, she suddenly asked - "is it because we are arranging squares of unit length inside shapes that we call the unit of area square-foot/ square-centimetre?" And she also figured out that the number of squares laid inside the shape is the area of the shape. She had encountered a big revelation on that day but it required more substantive talk around squares now to conclude this learning of area. This session would have easily lasted more than an hour.

Connecting area and perimeter formulae to unit squares

After a few days, I started the conversation by making a rectangle with equal sized small square pieces. In the meantime, in school they had done lots of practice questions on applying the formulae for area and perimeter for rectangles and squares; there was also a firm belief that writing the correct formulae was very important before one solved the question. I had made a rectangle with dimension 6×8 with small square tiles in two colours (blue and orange) and her task was to find the perimeter and the area of the rectangle (Figure 3).

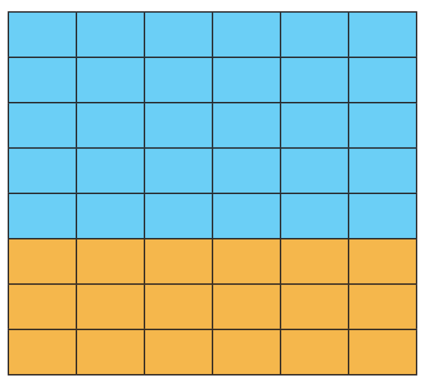

Figure 3

By now M had learnt, both through some work at school and at home, that for area one just needed to count the number of squares inside and for perimeter, she has to add-up the lengths of the sides of the given shape. We agreed to use the side of the squares as unit length and thereby the square itself having unit area. She found the area of the rectangle by counting the number of squares, in this case 48 unit squares. There was some hesitation in calling it square units and she would have preferred it as square metres, square centimetres or square feet. I intervened and told her that since we do not know the exact length of the squares, we can call it 1 unit, same as we had called the strip of paper for measuring length as 1 unit. She reluctantly agreed. Next, she found the perimeter by counting the sides of the unit squares, carefully counting each square in the boundary of the rectangle only once, which was found to be 24 units.

Having realized what M had done, I pushed her to use the formulae that she had learnt in school to see if she would get the same area and perimeter. She found the length and the breadth of the rectangle by counting the edges of the small squares that appeared along the length and breadth respectively. She correctly found these to be 6 and 8 respectively. The area formula for rectangle was l × b and putting these values gave her 48 unit squares. However, putting these values in the perimeter formula which was 2(l + b) led her to 28 units as the perimeter. Now she was confused. I left her with the task of figuring out which of the two results of perimeter was correct. She once again checked if she had counted the length and the breadth of the rectangle correctly by slowly moving her fingers over the squares in the boundary of the figure (see figure 3). Then she checked the perimeter by doing the same thing slowly so that she could understand the reason for the discrepancy. And of course in a little time she found that while she attempted to find the perimeter by counting the edges of the squares in the boundary for perimeter, she was counting the squares in the corner only once. However, for finding the length and the breadth of the rectangle, each square in the corner was counted twice, once towards the length and the other time towards the breadth. She resolved this conflict, indicating that the squares in the corner will need to be counted twice as they contributed to the length as well as to the breadth (See Figure 4 for the corner squares marked by a cross).

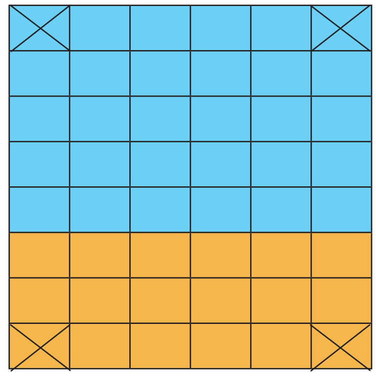

Figure 4

I further wanted to see if she could now see the reason for multiplying the length and the breadth of a rectangle to find the area of the rectangle. This was not easy. I kept prodding her to tell me some easier way of counting the number of squares in the rectangle. Unfortunately, she did not see the rectangle as being composed of 8 groups of 6. It took some time for her to recognize that she can count the squares by thinking it as 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6 or 6 × 8.

Exploring relationships between area and perimeter

Then we thought of rearranging the rectangle by removing some of the squares from the bottom to the side. We started doing this by trial and error but it was taking a lot of time and we were not sure how many rows we have to move to the columns to make it a rectangle with a different dimension. Eventually, we had to find some factors for 48. We found 12×4 as a good option and we arranged the new rectangle, as below (Figure 5). M once again found the area and perimeter of the new rectangle, this time without much difficulty. The area now remained the same but the perimeter changed to 32 units. She could see that since we have not changed the number of squares, but only rearranged them, the area should not change. But the fact that for the same area of the rectangle, its perimeter could change was surprising. She also thought that the difference between perimeter and area should lead to some pattern, probably a constant difference was in her minds, which was also falsified.

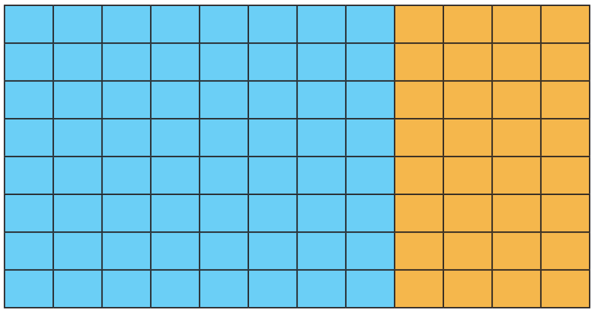

Figure 5

Then it was her turn to rearrange the squares to form a shape. Given the newly developing understanding of area and perimeter, she was no longer constrained to reorganize the small squares into a rectangle or square. She made an irregular shape, like the following (Figure 6), and she decided to remove one square, marked by Y. She was clear that the area had decreased by 1 as she had removed one square, but was bewildered by fact that the perimeter remained the same 32 units. Further, she removed another square marked by X, reducing the area by one more, but to her surprise the perimeter remained the same, once again. But this time, she was able to say why it was the case. She could articulate that the two side-lengths which were lost by removing X were compensated by the two new side-lengths which were opened-up by the removal.

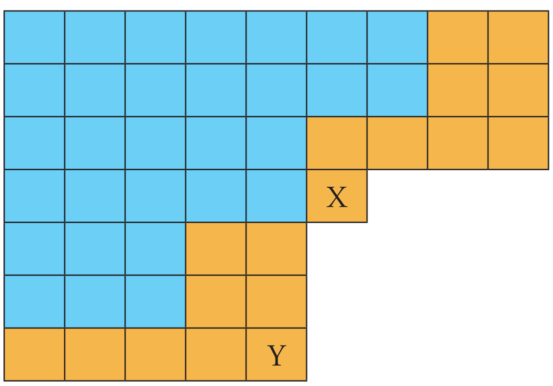

Figure 6

Final thoughts

This was the end of our conversation related to area and perimeter. Looking through the examples that were constructed during this conversation, we realized that we had constructed rectangles with the same area but different perimeters, irregular shapes with the same perimeter but different areas. M engaged and persevered with these tasks for a long time and tried to resolve the conflicts that emerged during the conversation.

However, there are still a lot of things which are unclear in the instructional sequence that I ended up developing as well as in M's understanding. Why did I not pursue seriously the idea of rectangular strips? Why did I distract her from using rectangles for measuring area and instead introduced my geometric ideas? What will happen if I filled the entire space using rectangles? What am I gaining by using a square of unit length? The fact is that we are looking for generalizable ideas, a core of mathematical thinking. When we use unit squares for measurement of area, the area of the square itself is 1 square-unit, much like 1 unit (or centimetre or metre) for measuring length, which has its own advantages. We need to find a unit which can help us measure length and breadth of the shape in a coordinated fashion and 1 square-unit helps us do that, and a rectangle-unit does not help us do that. We can now use this unit for measuring areas of different kinds of shapes and even in cases where the lengths are integers as well as fractions (will need to use ideas of fractions and decimals for this). I am not sure if M understands this and I did not give her enough time to explore this. I did try to talk to her about this after a few more days and showed her how the rectangle itself can be used to mark a square but it did not excite her much. So, I will perhaps postpone this to next year.

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