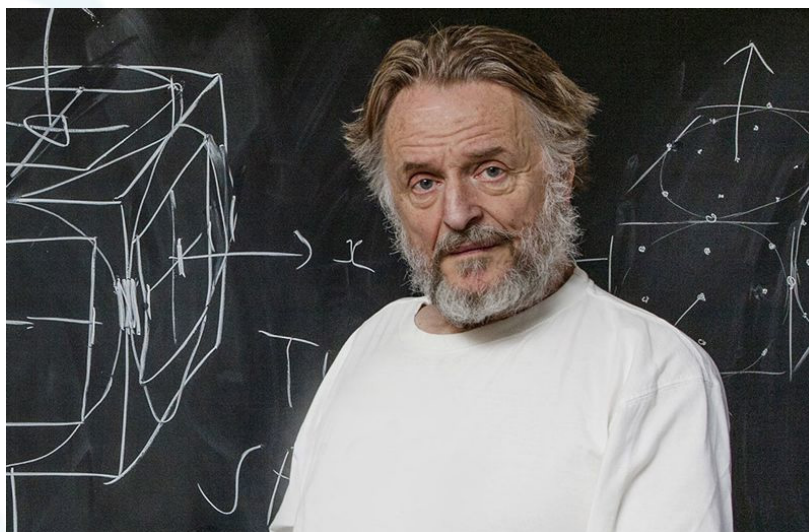


The Mathematical Artist of Play: A TRIBUTE TO JOHN HORTON CONWAY

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John Horton Conway at Princeton University in 2009.

Credit: Princeton University, Office of Communications, Denise Applewhite.

A Maverick Mathematician

In these strange and troubled times, when the world is reeling under an infectious disease and a quarter million have died from Covid-19 (as of May 1, 2020), it is perhaps inappropriate to mourn the death of a single individual who died of complications caused by the coronavirus. However, for the world of mathematics, this individual was very special indeed. **John Conway** stood out among the community of mathematicians, a legend in his lifetime.

Keywords: Nim, surreal numbers, combinatorial games, Game of Life, sphere packing, knot theory, Leech lattice, Conway Group, mathematician

Stephen Miller, a mathematician at Rutgers University, said: “Every top mathematician was in awe of his strength. People said he was the only mathematician who could do things with his own bare hands.” Conway liked to start from first principles, with very few concepts and build mathematical edifices. He loved to play games, preferably silly children’s games, all day long. He would spend whole summers going from one mathematics camp to another, one for middle school children here and one for teenage students, and in all of them, he would play games with children, pose and solve puzzles. He would carry all sorts of things with him: decks of cards, dice, ropes, coins, coat hangers, sometimes a Slinky, even a miniature toy bicycle. These were all props he would use for explaining ideas, though Conway insisted that they were more for his own amusement.

Unlike most other mathematicians, Conway was always obsessed with seemingly trivial concerns. He would be constantly factoring large numbers in his head. He could recite more than a 1000 digits of π from memory. He developed algorithms you could use to calculate the day of the week for any given date (in your head!), to count the number of steps while you climb stairs without actually counting, . . . But importantly, Conway claimed that it was such thinking that led to his mathematical research, and his colleagues at Princeton agree.

If anyone has ever approached all mathematics in a spirit of play, it is Conway. He also led the world in the mathematics of playing games.

Games as Mathematics

Consider *Nim*, a two player game, where there are a number of heaps of sticks between them. It is a turn based game. A player chooses one of the heaps, removes some non-zero number of sticks from it, then the other player gets her turn. When it is a player’s turn to move, and there are no sticks left, he loses; that is, the last one who makes a move wins. Here is the question.

Starting from any initial configuration of heaps, does one of the players have a *winning strategy*: a way to play in such a way that no matter what moves the other player makes, she is assured of a win?

How do we analyse such games? Suppose you are faced with the situation (1,1), when there are two heaps, each with one stick. This is a losing position for you, since no matter which heap you reduce (to empty), the opponent can move and then you have no move. Now, (2,2) is also losing since no matter what move you make, the opponent can copy that move and bring you either to (1,1) or to the empty configuration. Thus, in general, we can argue that the Nim position (m, n) is winning to a player iff $m \neq n$.

But this means that (m, m, n) is winning since you can remove the third heap entirely and present a losing position to the opponent. Similarly (m, m, n, n) is losing: we can consider it as $(m, m) + (n, n)$ or $(m, n) + (m, n)$, consisting of two subgames. Note that player II has a *copycat* strategy in the other subgame and hence cannot lose.

As it turns out, this operation $+$ yields a **group** structure on these games, where every element is its own inverse. (Comment. It is known that such a group is necessarily abelian, i.e., commutative.) The mathematical study of such bipartisan games (where both players have identical moves at any configuration) leads to very interesting combinatorics and algebra. The game of Nim was solved by Bouton in 1902 [2]. For tasting the mathematical adventure of both playing such games and analysing them, there is no better introduction than the book *Winning Ways for your Mathematical Plays* by Conway, Berlekamp and Guy [1].

Conway is credited with founding the area of *Combinatorial Game Theory*, a rich and beautiful subject of mathematical study. To give you a teaser, I have already mentioned that you can place an abelian group structure on games. There is a distinguished subgroup of games



John Horton Conway.

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called *numbers* which can also be multiplied and which form a field: this field contains both the real numbers and the ordinal numbers. In fact, Conway's definition generalizes both Dedekind cuts and von Neumann ordinals. All Conway numbers can be interpreted as games which can actually be played in a natural way; in a sense, if a game is identified as a number, then it is "so well understood that it would be boring to actually play it!" Conway's theory is deeply satisfying from a theoretical point of view, and at the same time it has useful applications to specific games such as Go. There is a beautiful microcosmos of numbers and games which are infinitesimally close to zero, and ones which are infinitely large. The theory also contains the classical and complete Sprague-Grundy theory of impartial games.

To my taste, the slim volume *Games and Numbers* by Conway [3] is nothing less than a masterpiece of mathematics. I recall seeing it in the library during my graduate student days, spending an entire day reading it right there, and rushing out to get it photocopied. Donald Knuth based a mathematical novel on these numbers [7]. Knuth called these numbers *surreal*, because

in this number field, every real number is surrounded by a whole lot of new numbers that lie closer to it than any other 'real' value does.

But to reiterate what was said earlier, playing games was intrinsically interesting to Conway, independent of all this algebraic structure. He created many games for people to play, especially for school children, and would spend enormous amounts of time on designing them. In fact, Conway is best known to the public for designing the **Game of Life**, a great boon to screen savers.

Game of Life

This is a very simple game. Take a sheet of grid paper, and keep a pencil and eraser with you. We have cells on the grid, each having eight neighbours. In this game, every cell can be *alive* or *dead* at any instant. There are (only!) two rules:

- A dead cell having exactly three live neighbours comes alive at the next instant; otherwise it stays dead.
- An alive cell that has two or three alive neighbours stays alive; else it dies.

Try out the evolution of the game starting from some random initial configurations. (You could do this using pencil and paper; or you could use the interactive website [9].)

Questions: If you start with any configuration of alive and dead cells, can we predict whether we would keep getting new configurations, or settle down to a specific configuration, or keep oscillating between some configurations? Pointing to a cell, can we figure out whether it will live for ever after some finite point in time? We can ask a variety of such questions. It turns out that there is no uniform algorithm to answer any of these questions. In fact, the Game of Life is exactly as powerful as the digital computer in a theoretical sense, and there is a variety of questions of this kind linking the game to computation theory and complexity theory.

In an interview in 2014 [4], Conway said he tinkered with the rules for “about 18 months of coffee times” before he arrived at such simplicity. He did not use any computers during this search either, he hand calculated the evolution of many configurations. This was typical of Conway, the search for extreme simplicity encapsulating almost universal capability, and doing it all ‘in the head.’

Mathematics as Play

Conway was born in Liverpool, England and went to study in Cambridge on a scholarship. In the 1960s, he worked on *sphere packing*. Suppose that you want to fit as many circles as possible into a region of the Euclidean plane. How can one do this? Divide the plane into one big hexagonal grid and inscribe the largest possible circle inside each hexagon. The grid, called a hexagonal lattice, serves as an exact guide for the best way to pack circles in two-dimensional space. In the 1960s, John Leech came up with a similar lattice for the most efficient packing of 24-dimensional spheres in 24-dimensional space.

Conway decided to study the symmetry group of the Leech lattice. It is called the Conway Group now. This led him to study the properties of similar groups.

In a paper in 1979, Conway and Simon Norton conjectured a deep and surprising relationship between the properties of the so-called monster group and that of an object in number theory called the j -function. The paper was titled *Monstrous Moonshine!* The monster group is a collection of symmetries that appear in 196,883-dimensional space. A decade later, Borcherds proved the conjecture, which won him the Fields medal in 1998.

Another area of mathematics in which Conway made an amazing contribution was **knot theory**, a branch of topology. Knots can be thought of as closed loops of string. A fundamental problem in the area is that of knot equivalence: can one apply finitely many allowed operations to obtain one from another? Mathematicians have different kinds of tests they can apply that act as invariants: if applying them to a pair of knots leads to different knots, the pair was different. One such test is called the Alexander polynomial, which is effective but not unique: the same knot could give rise to different Alexander polynomials. Conway fixed this, leading to what is known as a *Conway polynomial*, a fundamental tool in knot theory now. Another interesting contribution of Conway was an arrangement of knots, akin to the periodic table, that makes their properties easy to study.

In collaboration with Kochen and Specker, Conway proved the so-called *Free Will Theorem* in quantum mechanics: in crude terms, it states that if you had the information about the states of every particle in the universe up to this point, you would not be able to predict what their states will be a second from now. (Stated in still cruder terms, it states that if human beings have free will, then so do all elementary particles.)

Overall, Conway was active in the theory of finite groups, knot theory, number theory, combinatorial game theory and coding theory, with some work also in geometry, geometric topology, algebra, analysis, algorithmics and theoretical physics. The vast range attests to Conway’s ability to work on pretty much any mathematically posed question.

Conway was at Princeton University for the last quarter century, interacting intensively with students and colleagues. His biography by Siobhan Roberts [8] is a deeply inspiring story.

Extreme Elegance

Conway was known for his obsession for reducing proofs to the simplest terms. In 2014, Karamzadeh even argued that Conway's proof of Morley's theorem is the 'simplest possible' proof. Conway and Shipman [5] developed the idea of **Extreme proofs**. They consider 'values' we attach to proofs: brevity, generality, constructiveness, visuality, nonvisuality, 'surprise,' elementarity, and so on.

Indeed, because at any given time there are only finitely many known proofs, we may think of them as lying in a polyhedron (in our pictures, a polygon), and the value functions as linear functionals, as in optimization theory, so that any value function must be maximized at some vertex. We shall call the proofs at the vertices of this polygon the extreme proofs.

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They go on to study 7 extremal proofs of the assertion that $\sqrt{2}$ is irrational, tabulating them and explaining the associated value functions.

Last Words

For me, a cherished memory is a 45-minute journey from Rutgers to Princeton, when Conway gave me a car ride. He mentioned three interesting problems in combinatorics during the ride and when we reached, showed me a card trick before I left. I listened to a lecture by Conway, which was on graph theory, where he walked in with a structure he had built with magnetic rods, and used it to pose problems that led to enumerative combinatorics. What he talked about in the lecture was original research, but he never published any of it. This was typical of John Conway: doing mathematics was always in a spirit of play, occasionally some theorems might be for publication.



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