Features

What Can We Construct? – Part 1

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The geometers of ancient Greece invented a peculiar game for themselves, a game called ***Construction***, whose objective is to draw various geometric figures of interest. We are permitted to use just two instruments: an *unmarked straightedge* (a 'ruler'), and a *compass*. Using these, we can draw a straight line through any given pair of points, and we can draw a circle with any given point as centre and passing through any other given point. (Oh yes, we also possess a pencil and an eraser, please do not feel worried about that!)

The familiar school geometry box includes more than just these two items; it also has a marked ruler (to measure lengths), a protractor (to measure angles), two kinds of 'set squares' (to draw a right angle, and to draw a 60° angle; using these, we can also draw a line parallel to a given line and passing through a given point), and a divider. But we are not permitted to use these items if we wish to play the game as per the rules laid down by the Greeks. That is, the only instruments available are the unmarked straightedge and the compass. (Note that in the subject of engineering drawing, we are permitted the use of all of these instruments, and more. But that is another matter altogether.)

Using just these two instruments, how far can we go? What can we construct, and what can we not construct? This now becomes a mathematical question of considerable interest.

You may ask why we call it a 'game.' But it is just that, isn't it? – a game between you (i.e., the geometer) and the subject of geometry itself, played according to a fixed set of rules. If you are able to construct the required figure, you win; else, you lose; geometry wins and keeps its secrets!

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There are other areas where games of a similar sort have been devised. For example, there is the charming and delightful game of origami. Here too, we have a fixed set of rules (e.g., we must not use scissors), and we are required to perform all our operations and make all kinds of intricate objects within the boundaries set by these rules. Another example is that of solving polynomial equations using only the elementary operations (addition, subtraction, multiplication, division, taking of powers and roots). We shall have occasion to say more about these 'games' in later articles.

The central question. Let us state the central question of the construction game more clearly. On the plane, mark two points O and A; take the distance between them to be 1 unit. Draw the infinite straight line through O and A. Think of this as the x-axis of the coordinate plane, with O as the origin and A as the unit point, with coordinates $(1, 0)$. Define a set **CR** as follows ('CR' for compass and ruler): **CR** consists of all real numbers x such that we can locate the point P with coordinates $(x, 0)$, using only a compass and an unmarked straightedge. (Another way of putting this: we can construct a segment of length $|x|$ using only a compass and an unmarked straightedge.) We refer to **CR** as 'the set of constructible numbers.' It is clearly a subset of the set of real numbers. But what is the nature of this set? Which numbers lie in it, and which numbers do not? Is there a simple way of deciding membership of **CR**? We shall explore these questions in this article.

Properties of the Set of Constructible Numbers

We can use the compass repeatedly to lay out multiples of the unit length on the x-axis, as many as we wish. This tells us that every integer is part of **CR**.

There is a simple ruler-and-compass construction by means of which we can divide any given line segment into any given number (a positive integer) of equal parts. (It uses the properties of parallel lines and similar figures.) From this, it follows that every rational number is part of **CR**.

We now prove a series of properties of **CR**. The first result is obvious, but the others may come as a surprise.

Theorem 1 (Closure under addition and subtraction)**. CR** *is closed under both addition and subtraction. That is, if a and b are elements of* **CR**, *then so are $a + b$ and $a - b$.*

Proof. To streamline the writing, let us suppose that $a > 0$, $b > 0$ and $a > b$. It should be clear that if we can separately construct line segments having lengths a and b, then we can also construct line segments having lengths $a + b$ and $a - b$ (see Figure 1). Here, we use the fact that we can transfer distances using a compass and an unmarked straightedge. $\square$

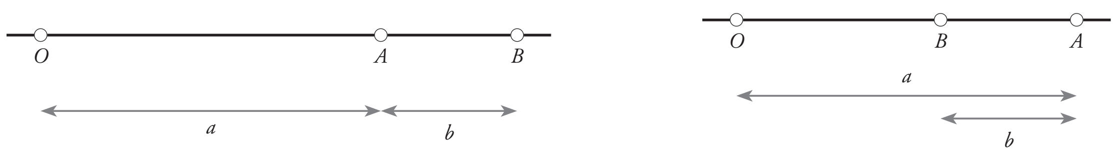
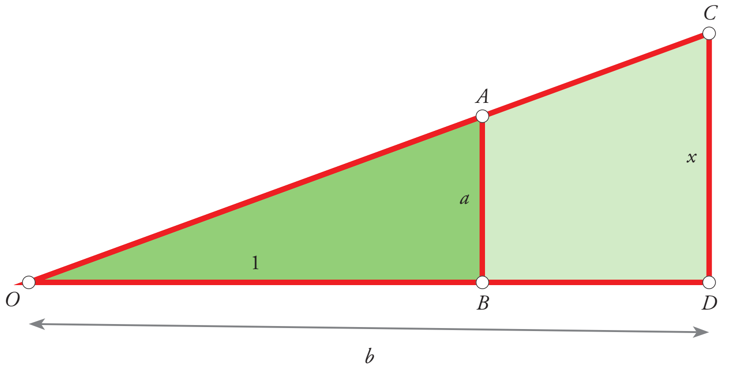


Figure 1.

Theorem 2 (Closure under multiplication). **CR** *is closed under multiplication. That is, if a and b are elements of* **CR**, *then so is ab.*

Proof. Let a, b be given positive numbers. We show two ways in which we can construct a line segment of length ab, starting with line segments of lengths a and b respectively.



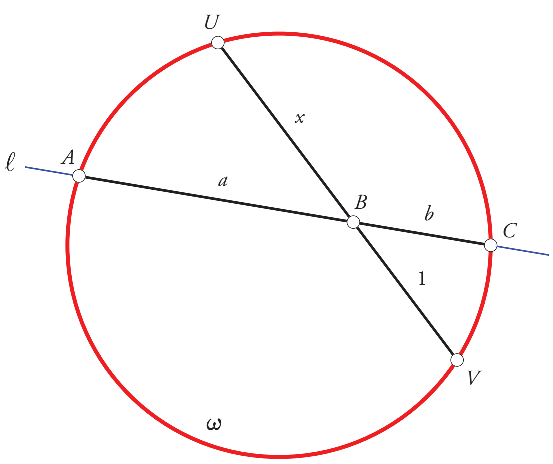
$$a : 1 = x : b,$$

$$\therefore\ x = ab$$

Figure 2. Here O, A, C are collinear, as are O, B, D, and $CD \parallel AB$.

The first method (see Figure 2) uses the fact that the number x satisfying the equation $a : 1 = x : b$ is $x = ab$. We use a pair of similar triangles to obtain the solution. Figure 2, which should be self-explanatory, gives the details. (In the figure, we suppose that $b > 1$; but that is obviously not a restriction. We have merely taken it so for convenience.)

The second method is based on the intersecting chords theorem of circle geometry. (Of course, here too we draw upon the properties of similar triangles.) Figure 3 gives the details.



Explanation. On a line ℓ, mark three points A, B, C, such that $AB = a$, $BC = b$. Locate a third point V, not on ℓ, such that $BV = 1$ unit (as defined previously). Draw the circle ω passing through points A, C, V. Extend segment VB beyond B till it meets the circle again at U. Let $UB = x$ units.

Figure 3. Using the intersecting chords theorem to do multiplication

Using the intersecting chords theorem, we get $a \cdot b = x \cdot 1$, and hence $x = ab$. □

Theorem 3 (Closure under division). **CR** *is closed under division by nonzero numbers. That is, if a and b are elements of* **CR**, *where $b \neq 0$, then a/b is an element of* **CR**.

Proof. We show two different ways (analogous to the above) by which to do the division.

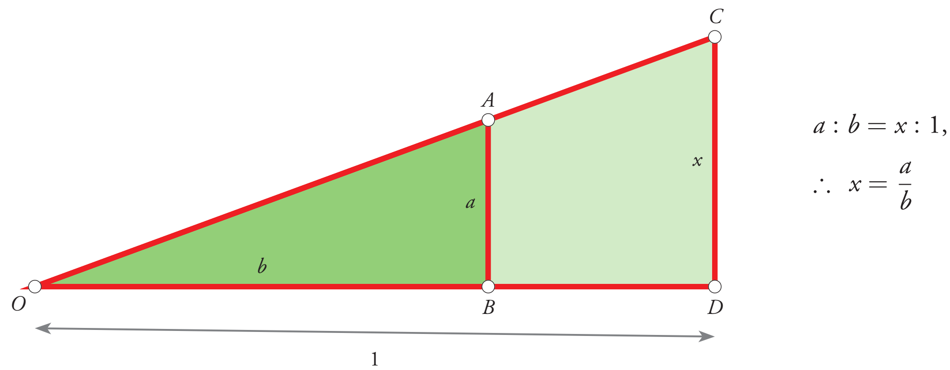
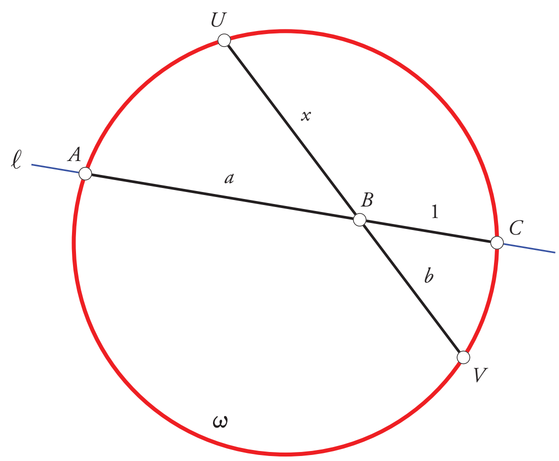


Figure 4. Here O, A, C are collinear, as are O, B, D, and $CD \parallel AB$.

Figure 4 shows the first way, using a pair of similar triangles. The figure has been drawn under the supposition that $b < 1$; but that is only for convenience and is not a restriction.

Figure 5 shows the second way, using the intersecting chords theorem.



Explanation. On a line ℓ, mark three points A, B, C, such that $AB = a$, $BC = 1$ unit (as defined previously). Locate a third point V, not on ℓ, such that $BV = b$. Draw the circle ω passing through points A, C, V. Extend segment VB beyond B till it meets the circle again at U. Let $UB = x$ units.

Figure 5. Using the intersecting chords theorem to do division

Using the intersecting chords theorem, we get $a \cdot 1 = x \cdot b$, and hence $x = \dfrac{a}{b}$. $\square$

Field structure. The fact that the set of constructible numbers is closed under addition, subtraction, multiplication and division by non-zero numbers should alert us to something highly significant. Any subset of the set of real numbers $\mathbb{R}$ that has at least one nonzero number and is closed under addition, subtraction, multiplication and division by nonzero numbers is an example of a **field**. Obviously, $\mathbb{R}$ itself is a field; but $\mathbb{R}$ has very many proper subsets of great interest which are also fields.

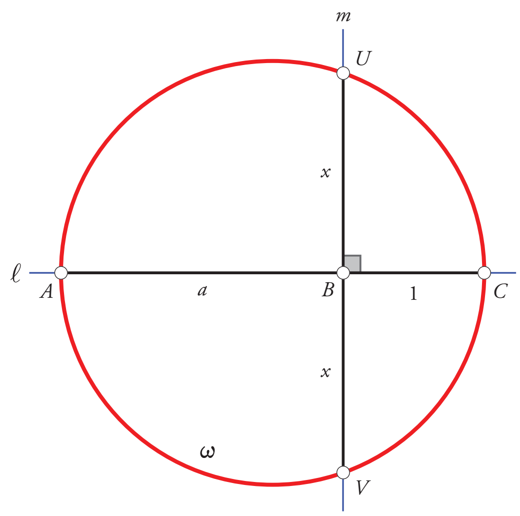
(Before proceeding, we must note that fields are abstract structures that are defined much more generally. We can have fields that have nothing to do with the real numbers. In this brief note, however, we only consider fields that are subsets of the set of real numbers.)

The simplest and most obvious example of a subset of $\mathbb{R}$ which is a field is the set of rational numbers. It is generally denoted by the symbol $\mathbb{Q}$. This field has the interesting property that it has no proper subset which is also a field. So, $\mathbb{Q}$ is the smallest possible field which is a subset of the set of real numbers $\mathbb{R}$. Or, stated more concisely: $\mathbb{Q}$ is the smallest possible subfield of $\mathbb{R}$.

This is easy to show. Let S be a subfield of $\mathbb{R}$; then S contains at least one non-zero number k. Since S is closed under division by nonzero numbers, it contains the number $k \div k$, i.e., it contains the number 1. Since S is closed under addition and subtraction, it contains every possible integer. Invoking again the fact that S is closed under division by nonzero numbers, we see that S contains every rational number. This means that $\mathbb{Q}$ is a subset of S. So there cannot be a subfield of $\mathbb{R}$ which is smaller than $\mathbb{Q}$.

Back to constructibility. We return to the study of the set of constructible numbers, **CR**. Using the above terminology, we see that **CR** is a subfield of $\mathbb{R}$. Is **CR** identical to $\mathbb{Q}$? The next result, which may come as a surprise, shows that **CR** is much larger than $\mathbb{Q}$.

Theorem 4 (Closure under the square root operation)**.** *The set of constructible numbers* **CR** *is closed under the square root operation. That is, if a is constructible, and $a > 0$, then $\sqrt{a}$ is constructible.*



Explanation. On a line ℓ, mark three points A, B, C, such that $AB = a$, $BC = 1$ unit. Draw the circle ω on AC as diameter. Draw a line m perpendicular to ℓ at B. Let m intersect the circle at points U, V. Then we have $UB = BV$. Let $UB = x$ units.

Figure 6.

Proof. See Figure 6. Using the intersecting chords theorem, we get $a \cdot 1 = x \cdot x$, and therefore that $x = \sqrt{a}$. □

All kinds of numbers .... Using Theorems 1, 2, 3, 4 in combination, we find that the set **CR** contains all kinds of numbers! Here is a small sample:

$$\sqrt{2}, \quad \sqrt{2+\sqrt{3}}, \quad \frac{\sqrt{3}+1}{\sqrt{5}+1}, \quad \frac{\sqrt{2+\sqrt{3+\sqrt{5+\sqrt{11}}}}}{1+\sqrt{5-\sqrt{2-\sqrt{3}}}},$$

and so on. It is hard to imagine any circumstance under which one may want to construct a segment whose length is the fourth number listed above! But the important point is that we **can** construct a segment with this length, if we wish to.

In a follow-up article, we shall consider some very famous construction problems which the Greeks had posed for themselves and which remained unsolved for an extraordinarily long period of time.

Appendix: Some Subfields of the Set of Real Numbers

Purely for the sake of completeness, we give here in the appendix some examples of subfields of $\mathbb{R}$.

Example 1: For our first example, we make use of the square root of 2. Let the set S be defined as follows

$$S = \left\{ a + b\sqrt{2} \ : \ a, b \in \mathbb{Q} \right\}.$$

That is, S consists of all numbers of the following form: a rational number + a rational multiple of $\sqrt{2}$. It is easy to see that S is closed under addition, subtraction, and multiplication. To see that it is also closed under division by nonzero numbers takes a little more effort. For this, we need to verify that if a and b are rational numbers, not both 0, then the following number

$$\frac{1}{a + b\sqrt{2}}$$

lies in S. To verify this, we use the old-fashioned technique of rationalisation:

$$\frac{1}{a + b\sqrt{2}} = \frac{1}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.$$

The crucial observation now is that the denominator of the last expression cannot be 0. This is true because $\sqrt{2}$ is an irrational number. (If $a^2 - 2b^2 = 0$, then it would mean that $\sqrt{2} = a/b$, which is not possible since $\sqrt{2}$ is irrational.) So we have managed to express $1/(a + b\sqrt{2})$ in the form 'a rational number + a rational multiple of $\sqrt{2}$.' The claim that S is closed under division by nonzero numbers is thus proved. Hence S is a subfield of the field of real numbers, as claimed.

Example 2: This is similar to the first example, except that we use the square root of 3 rather than the square root of 2. Let the set T be defined as follows

$$T = \left\{ a + b\sqrt{3} \ : \ a, b \in \mathbb{Q} \right\}.$$

By going through the same steps as earlier, we may check that T is a subfield of the field of real numbers. The crucial point is that $\sqrt{3}$ is an irrational number. This ensures that we can express $1/(a + b\sqrt{3})$ in the form 'a rational number + a rational multiple of $\sqrt{3}$.'

Infinitely many such subfields can be constructed, using the square roots of other positive integers which are not perfect squares.

We will say more about this topic in a follow-up article.

References

1. Wikipedia, "Straightedge and compass construction," https://en.wikipedia.org/wiki/Straightedge_and_compass_construction.

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