Mathematical Doodling using the what-if-not approach

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There are many ways of generating new problems. This paper proposes one of them, the "what-if-not" approach. The write-up shares how this approach was used to do explorations and create new problems. Not all the generated problems could be solved but the experience was indeed enriching and overwhelming.

Problem-solving, which has always been an important part of learning mathematics, received considerable attention after it was recognised as a route for promoting mathematical thinking in the Position Paper on Teaching of Mathematics (NCERT, 2005). For problem-solving to flourish in its true spirit, two ingredients are essential: adequate knowledge and skill to solve the problems and the acumen to generate good meaningful problems.

How do we pose new problems and what must be done to generate them? Often, teachers feel that problem solving and problem posing are "out-of-the-syllabus" activities and regard them as 'extra' work. We propose problem posing as an act of extension to the existing textbook problems so as to let children and teachers build a deeper connection with the textbook. In this article, we illustrate an approach that can be adopted for generating new problems from the existing textbook problems. We are suggesting a way through which students can be involved in making and solving many problems generated from the basic problem in the textbook. They get to identify the underlying conditions that define a particular problem and then change these conditions one-by-one to create more problems. We believe that in this way, students get tempted to challenge textbook problems and feel a desire to know new concepts.

We present the 'what-if-not' approach that can be used for generating problems from the existing ones. The 'what-if-not' strategy opens avenues for challenging, observing, creating new situations and delving into newer ideas. When people create their own problems they also get persuaded to solve them, thus begins

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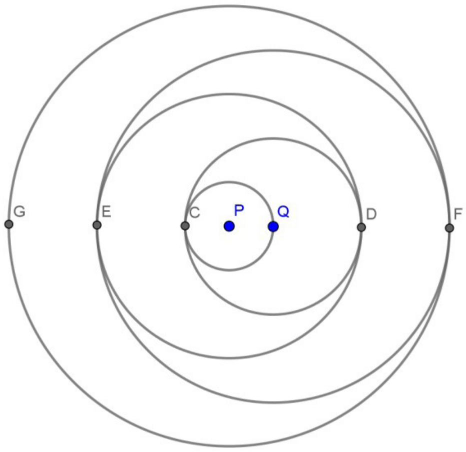


P and Q are alternate centres
of the circles produced with radii
in AP: 2cm, 4cm, 6cm, 8 cm, ...

Figure 1a

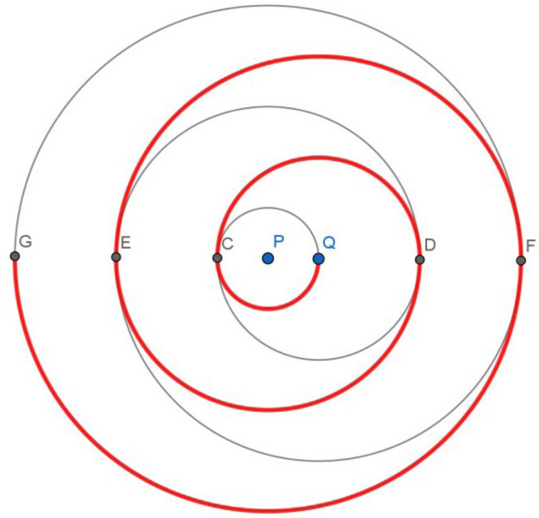


Figure 1b

a personal journey of mathematical thinking. People look out for patterns, make conjectures, deduce logic and may propose generalisations and even proofs.

The what-if-not approach

The 'what-if-not' approach proposed by Brown and Walter (1972) is based on identifying the key attributes of a problem, making a note of the attributes, modifying or altering each attribute one at a time to make new problems; in turn, also finding newer ways of solving them. Each alteration from the original problem offers scope for generation of a new problem.

It works like this. Every problem has some conditional statements. For example, the statement, *The product of two consecutive positive integers is …* rests on three conditions:

Condition 1: The numbers are positive integers

Condition 2: The numbers are consecutive

Condition 3: The numbers undergo an operation of multiplication.

In any mathematical learning act, recognising these conditions is an essential step. The mathematical journey begins when these conditions are challenged.

That is, What if:

- The first condition is changed? The selected numbers are not positive integers, will the result be any different?
- Condition 2 is changed? What would happen if the positive integers differ by two?
- Only Condition 3 is altered? Instead of multiplying, some other operation is performed?

How would the results get affected? We used the 'what-if-not' approach on a problem from the Class X, NCERT textbook.

A spiral is made up of successive semi-circles, with centres alternately at P and Q, starting with centre at P, of radii 2 cm, 4cm, 6 cm,… as shown in the figure. What is the total length of such a spiral made of 13 consecutive semi-circles? (Class X, NCERT, 2006)

The above NCERT problem was attempted on GeoGebra. Taking P as centre, a circle C1 of radius 2 cm was created. Then, the next circle, C2 was created taking point Q as centre and radius 4cm. Circles C3, C4 and henceforth were created by alternating the centres P and Q and increasing the radii each time by 2cm (Figure 1a).

From the network of the circles created, a spiral, marked in red, emerges on joining the points of contact of the circles. The first leg of the spiral is the semi-circle of the first circle C1, from Point Q to C. The next leg of the spiral emerges from the point of contact of Circle C1 and C2, at point C. We traced the semi-circle of circle C2 from point C to point D. Continuing this process, the points of contact of two consecutive circles serve as the emerging points for the on-shoot of the next leg of the spiral (Figure 1b).

There are three essential conditions which led to the semi-circle spiral:

Condition 1: The radii of the circles are in Arithmetic Progression. In the given problem, the radius of the first circle is 2 cm and that of each subsequent circle increases by 2 cm. In terms of the conventional nomenclature used to represent Arithmetic Progressions, we can say that the initial term ('a') and the common difference ('d') are the same.

Condition 2: The spiral is made up of successive semi-circles.

Condition 3: The centres of the circles alternate.

The "what-if-not" approach rests on altering the underlying conditions of a problem one by one. The conditions can be modified and/or negated one at a time. Then, based on each alteration, attempts to solve the new problem are done.

We demonstrate the work done by us using this approach, with the help of a dynamic software (GeoGebra). Each of the above conditions was altered to capture a new view. New situations were created and new patterns emerged. However, we confess, we were not successful in finding solutions to our created tasks. Nevertheless, the attempts gave us meaningful insights into what it really means to be a mathematician. Thus, we add a caveat. At some times, you may find it difficult to solve the newly generated problem, but you would be happy to have created another problem. There could also be times when you may not be able to pose another new problem but remember that that is learning in itself. The idea is to generate more problems in a connected way; and so we present our journey.

Altering only Condition 1: What if, the initial radius and the difference between the consecutive radii are not the same (if we have an AP in which 'a' is not equal to 'd') Is it possible to get semicircular spirals?

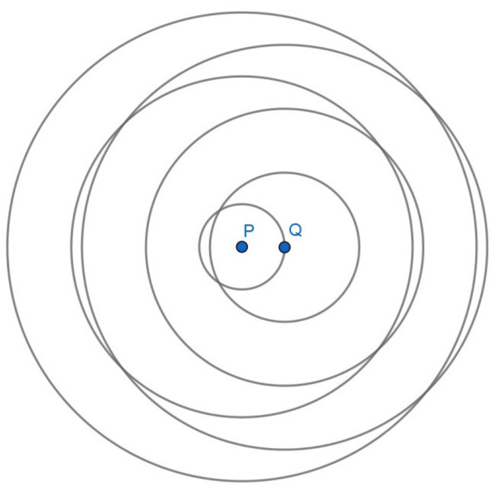


Figure 2a

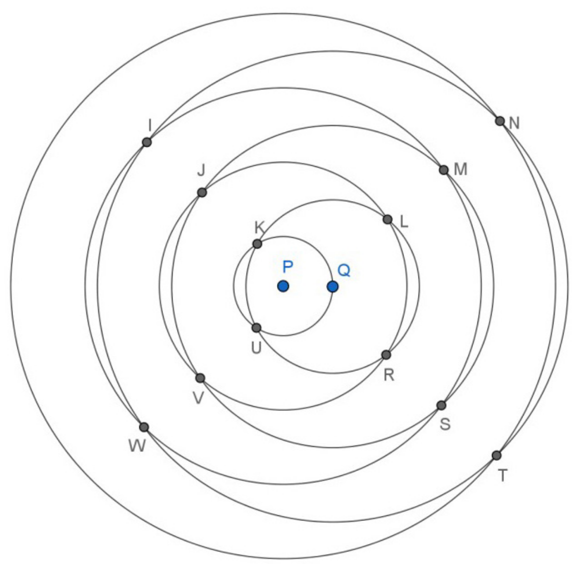


Figure 2b

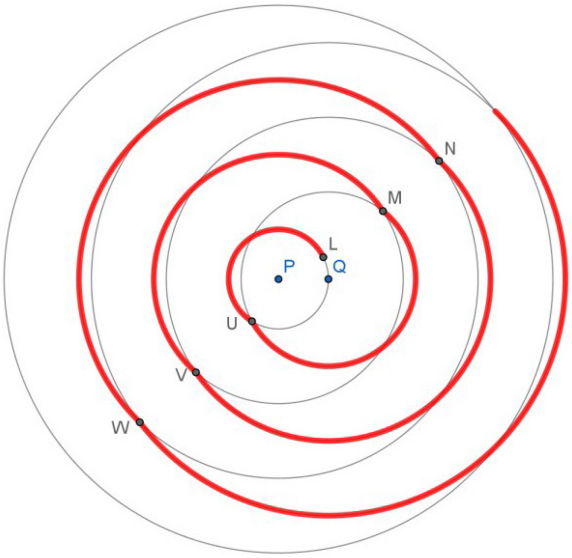


Figure 3a

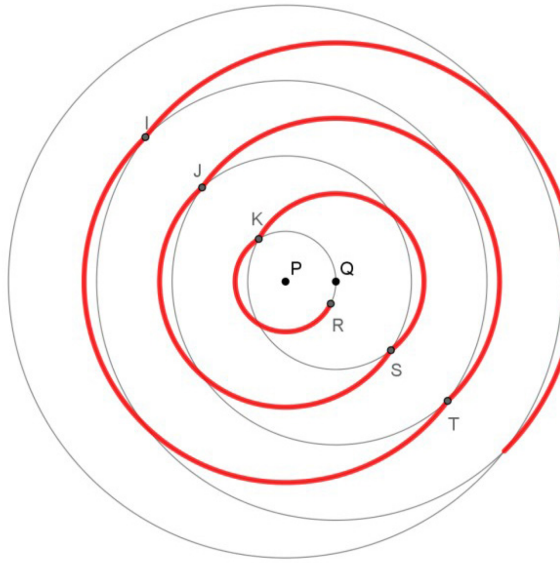


Figure 3b

Two sub-conditions emerge on altering Condition 1:

Sub-condition 1.1: The radius of the first circle is greater than the difference between the radii of the subsequent circles. That is, 'a' is greater than the common difference, 'd'. a > d

Sub-condition 1.2: The radius of the first circle is less than the difference between the radii of the subsequent circles. That is, 'a' is less than the common difference, 'd'. a < d.

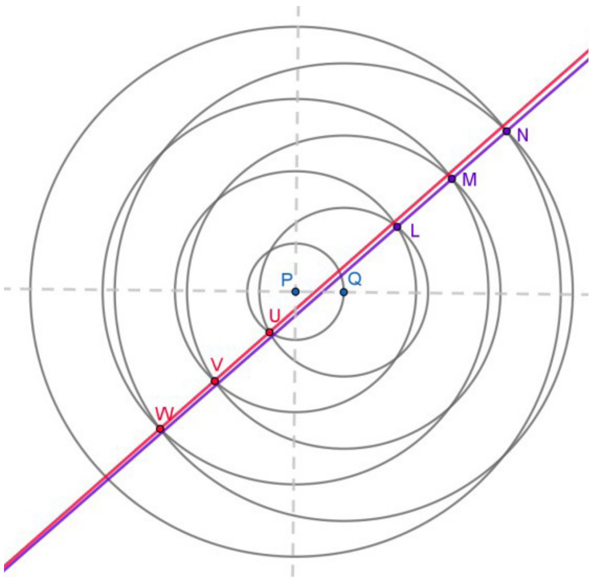
Consider Sub-condition 1.1: To illustrate the construction geometrically, we took the radius of the initial circle as 4 units and the difference between the radii of each subsequent circle as 3 units. Figure 2a emerges.

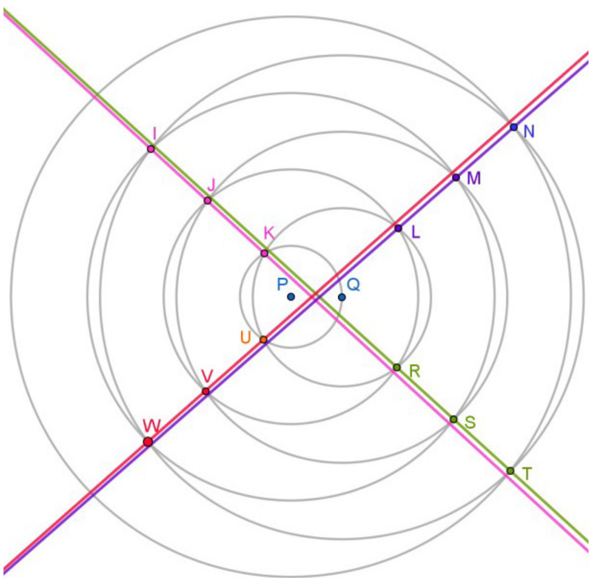
At a glance, one could see that the corresponding circles intersect each other at two points. The first two circles intersect each other at points K and U, the second and third circles at points L and R, and so on (Figure 2b). As done earlier, we took the points of intersection for making the spirals. Each point of the intersection of circles was taken as the emerging point for the next leg of the spiral. We got two pseudo-spirals, one clockwise and other anticlockwise (Figure 3a and Figure 3b).

In addition to getting the spirals, we also observed a pattern. It seemed that the points of intersection of the circles (i.e., points N, M, L, U, V, W and points I, J, K, R, S, T) are respectively collinear. Thus, we made a hypothesis, "Lines passing through the points of intersection of the circles intersect each other." In other words, our visually-based hypothesis was, "Points N, M, L, U, V, W and points I, J, K, R, S, T are collinear and that the lines passing through these points would intersect at the origin."

The hypothesis was made only on the basis of our visual perception and it soon got refuted when we joined the points on the dynamic tool. We got four lines which neither intersected nor were they mutually parallel. However - the points lying in one quadrant were seen to be collinear-the conjecture is yet to be proved. (Please note, the four quadrants were made taking the x-axis as the line on which the centres of the circles P, Q lie and the y-axis was the perpendicular line passing through the centre of the first circle, i.e., through point P.)

Alas, we couldn't go any further, even though we could sense the presence of some hidden



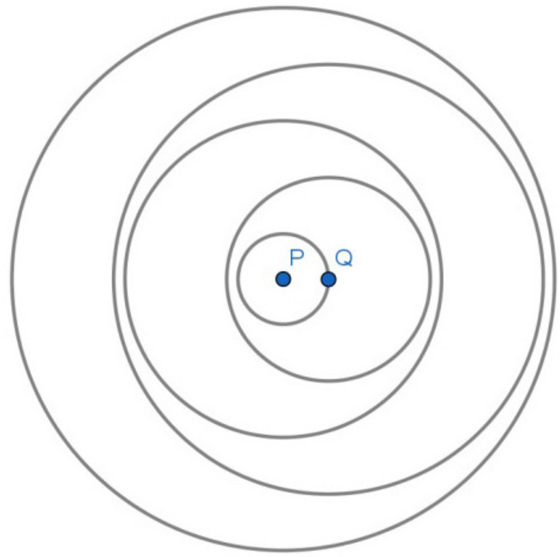


mathematical gems. Perhaps, somebody will be able to draw out more sophisticated conclusions.

Sub-Condition 1.2: Radius of the first circle is less than the difference between the radii of the subsequent circles.

To construct circles based on the above sub-condition, we took the radius of the first circle as 4 units and the difference between the radii of the consecutive circles as 5 units.

Observation: The circles do not intersect each other so it was not possible to make spirals.



P and Q are the respective centres of the circles having radii in AP: 4, 9, 14, 19, 24,...

Next, we tried altering Condition 2.

Altering Condition 2: What if a spiral is made not of successive semi-circles.

We altered the semi-circles and replaced them with semi-polygons. To make things simpler, we started with a regular quadrilateral, i.e., the square. We were now interested in making uniformly growing squares whose centres would alternate and semi-squarish spirals could be made.

In a square, the centre is fixed but to make the growing squares we had two options:

- to consider the distance from the centre to vertices in A.P. i.e., increase the lengths of half-diagonals in A.P. or
- to consider the perpendicular distances from centre to the midpoints of the edges in A.P. i.e., increase the apothems in A.P.

Both sub-cases led to two ways of making the semi-square spirals:

Sub-Condition 2.1: Half-diagonals increase in AP. Consider the distance between the centre of the first square and its vertex as the initial distance, and subsequently increase the length of every half-diagonal by the same

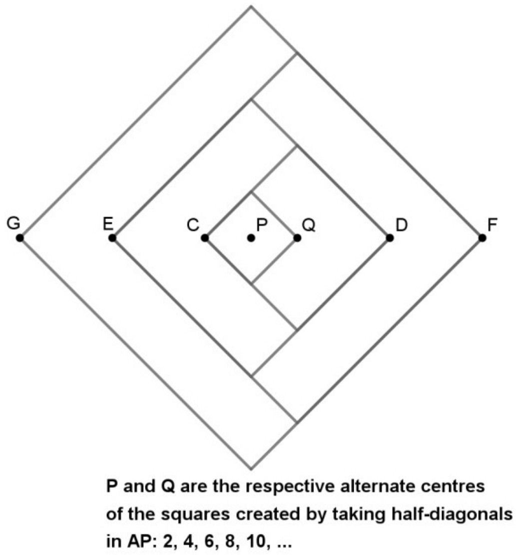


Figure 4a

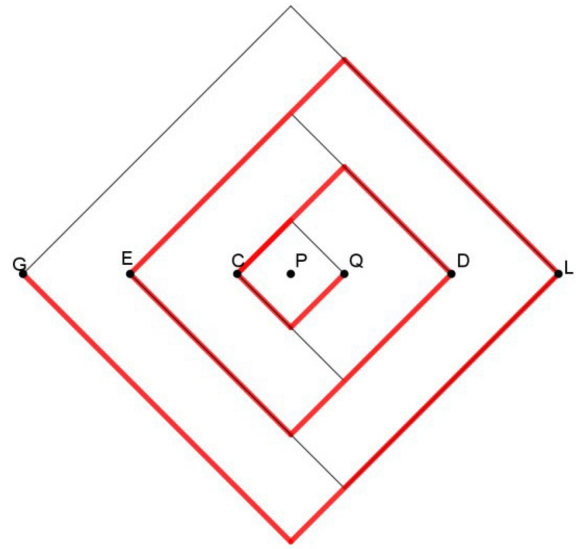


Figure 4b

magnitude. While altering Condition 2, the original Conditions 1 and 3 were kept intact. Thus, the lengths of the half-diagonals of the subsequent squares were in Arithmetic Progression with a = d, and the centres of the squares alternated. The semi-squarish spiral depicted in the Figures 4a and 4b emerged.

Sub-Condition 2.2: Apothems increase in AP.
Drop a perpendicular from the centre of the first square to the midpoint of a side and consider this as the initial distance. For each subsequent square, the length of the apothems will increase by same magnitude. The following square-spiral emerged (Figures 5a and 5b).

Similarly, one can explore more semi-polygonal spirals such as semi-pentagonal-spirals, semi-hexagonal-spirals, using GeoGebra.

And, finally altering Condition 3: What if the centres of the circles do not alternate?

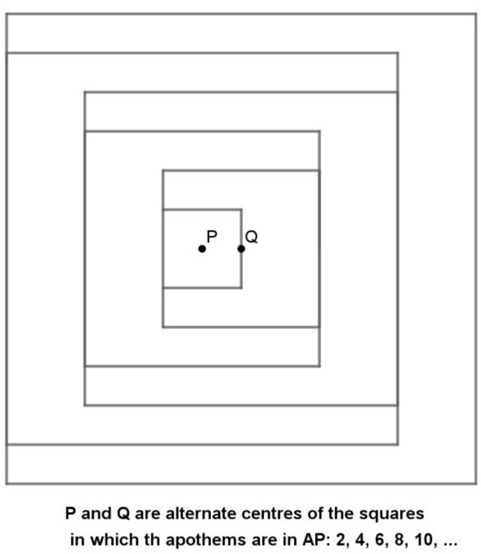


Figure 5a

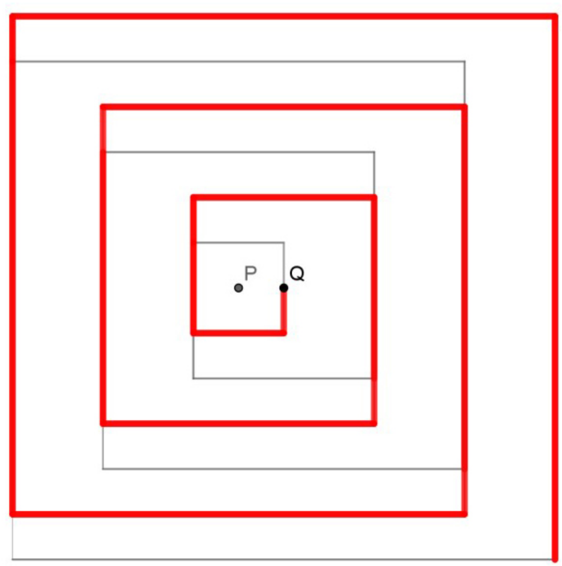


Figure 5b

If the centres do not alternate, and the other conditions remain unaltered, it will produce consecutive circles and no spirals.

Ending Remarks

While doing this exercise we wondered, why have we not opened the gates to problem-generation? There could be many reasons, one among them is that the curriculum makers or teachers frequently do not appreciate problem generation. What we mean is that teachers only value 'neat' problems which proceed on formal procedures, culminating in neater answers. Problem generation should not be pursued with a mindset of promoting a neatly framed problem which would always pave formal structures to the solution. What is needed is an appreciation to value intuition that is built on logical and justifiable observations. We need to build an acumen to make hypotheses and conjectures in a structured way without being bothered, at least at that moment, of generating proofs.

Nowhere in this activity are we claiming that we were led to solutions or any formal theorising. Lest we lose sight of the larger picture, the work shared by us only provides a glimpse on how problems can be generated or expanded from routine textbook problems. Each expanded problem may not have an answer . Often engagement with the tasks may be very different from that expected. Asking relevant questions, making conjectures based on perceptions and generating problems invokes a spirit of inquiry, a desire to explore. We propose to open avenues for discussions, explorations, observations, visualisations, patterns and generalisations to the extent possible. While trying to work on the alternative conditions, we were guided by our intuitions and visual connections. The excitement came from our thrill of observing what emerged by changing the conditions. We admit we are not experts in mathematics, but we dare to say that you needn't be an expert to let your curiosity pull you in. We were doodling mathematically. We were seeing newer problems which may not be 'neat' in a true sense but they do hold the potential of becoming sophisticated ones.

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