Upper Primary

ClassRoom

# Triangles to Tetrahedrons and beyond…

The seed-idea of this article came from an activity from an upper primary math textbook and the modification in a subsequent edition. Students were asked to find the midpoints of the sides of an acute isosceles triangle and join them to form four smaller triangles, and then fold the triangles up to a tetrahedron. An equilateral triangle replaced the isosceles one in the subsequent edition. What caused this change? Wouldn't any triangle generate a tetrahedron? This initial exploration revealed something unexpected and the findings had an eerie resemblance to a known result. Further discussions with more math-friendly minds watered and added subsequent layers to this exploration and took it to a newer dimension – figuratively and literally! If a perpendicular is dropped from the apex (which is the top vertex of the tetrahedron where all three vertices of the triangle meet) to the base, where will the foot of this perpendicular be? For an equilateral triangle, it is the centre of the base but would it ever be coincident with any of the triangle centres, i.e., centroid, circumcentre, incentre or orthocentre of the base for other triangles? We will investigate these.

This Low Floor High Ceiling (LFHC) investigation begins by considering a neglected question on types of triangles. Then it explores a particular property that helps us classify triangles. After that, we zoom into one class of triangles and transition into 3-dimensions using nets. The nets, and the solids, in turn, generated more questions as well as helped in tackling them.

The first task considers types of triangles. How many are there?

**Task 1:** If we consider the sides, a triangle can be isosceles or scalene and some isosceles triangles can be equilateral too. Angle-wise, a triangle is acute, right or obtuse.

*Keywords: Triangle, tetrahedron, constraint*

a. How many types of triangles are there when both side-wise and angle-wise criteria are considered?

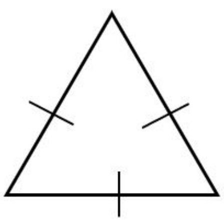
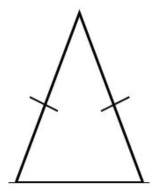
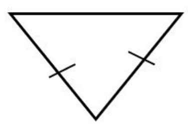
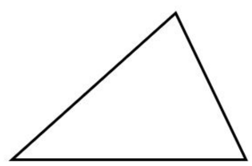
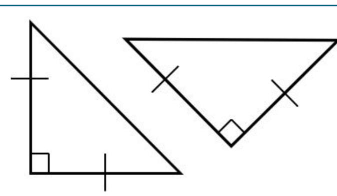
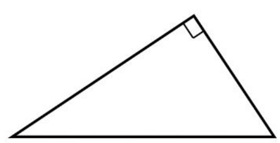
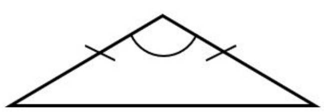
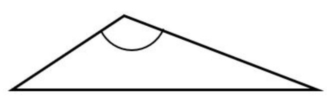
b. For each of the above categories, find at least two possible angle combinations, e.g. 30°–60°–90° and 40°–50°–90° for right scalene. Are there any types where only one angle combination is possible? If so, which one(s)?

c. Consider two acute isosceles with angle combinations (a) 20°–80°–80° and (b) 80°–50°–50°. Compare the unequal side with the equal sides. Find one more triangle with angle combination like (a) and another one like (b). How are the triangles like (a) different from those like (b)?

d. Consider the two groups of acute isosceles triangles (a) and (b) in the previous problem. Which type of triangle separates these two groups? Which type of triangle separates the obtuse isosceles from the acute ones?

**Teacher Note:** There are eight types of triangles as follows: three types of scalene – acute, right and obtuse, and four types of isosceles – equilateral, acute (with a different 3rd side), right and obtuse. Out of these, equilateral and right isosceles form similar class of triangles with 60°–60°–60° and 90°–45°–45° angle combinations respectively. Acute isosceles can be of two types depending on the equal sides being (a) longer or (b) shorter than the unequal one. This comes out very well if one tries to make triangles with just 10 matchsticks. The possibilities are 2–4–4 and 3–3–4 illustrating the two cases (a) and (b). The equilateral separates these two cases (a) and (b). So, there are five kinds of isosceles and they can be characterized by the angle θ between the equal sides: (i) $0° < \theta < 60°$ or the type (a) acute, (ii) $\theta = 60°$ i.e. equilateral, (iii) $60° < \theta < 90°$ or the type (b) acute, (iv) $\theta = 90°$ i.e. right, and (v) $90° < \theta < 180°$ or obtuse. Table 1 includes all eight types of triangles.

The second task was seeded by the textbook that changed the triangle from acute isosceles to equilateral in subsequent editions. So, let us explore what happens for all eight types of triangles.

| **Table 1** | Equilateral | Isosceles | Scalene |
|---|---|---|---|
| Acute |  | Type a <br>Type b  |  |
| Right | |  |  |
| Obtuse | |  |  |

**Task 2:** Consider any ΔABC. Find the midpoints D, E and F of the sides BC, CA and AB respectively. Join DE, EF and FD.

a. Will ΔAEF, ΔBFD, ΔCDE and ΔDEF be the four faces of a tetrahedron (see Figure 1)?

b. If not, find the criteria for not getting a tetrahedron.

c. Is there any border-line case? Explain.

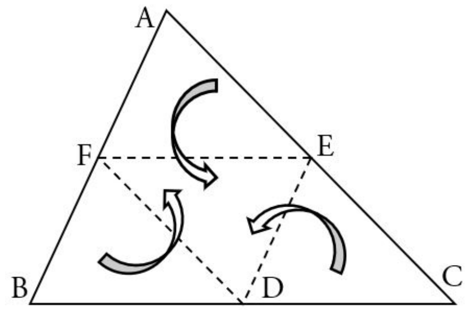



Figure 1

**Teacher Note:** Interestingly, not all triangles can be folded to a tetrahedron. The acute angled triangles (both isosceles and scalene) fold up and meet at a point to form a tetrahedron. But the obtuse angled triangles do not because two of the folded edges stay apart. The right triangles may create some confusion since the folded edges do match up (unlike obtuse) but no solid is formed (unlike acute). The point at which the folded sides meet is on the plane of the triangle so no solid is formed.

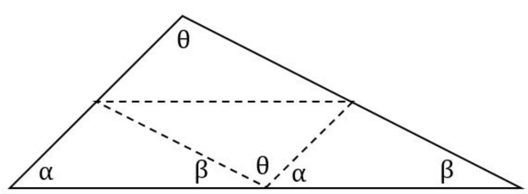



Figure 2

A closer inspection of the angles formed at the midpoint of the longest side reveals the cause (see Figure 2). In an obtuse triangle, $\alpha + \beta < 90°$ and $\theta > 90°$. Naturally α and β cannot cover all of θ (see Figure 3). Therefore, there is a gap between the folded edges. In the case of right triangles, $\alpha + \beta = 90° = \theta$ i.e. α and β cover θ exactly. So, the folded triangle flattens out with the edges meeting perfectly. Only in the case of acute triangles, $\alpha + \beta > 90° > \theta$. Therefore, α and β not only cover all of θ but actually overlap a bit. When the edges are put together to avoid the overlap, we get a solid, a tetrahedron, with a definite height.

So, a tetrahedron can form if and only if (iff) $\alpha + \beta > \theta$. It is worth noting that this angle inequality for a tetrahedron is very similar to the inequality involving the sides of any triangle ($a + b > c$, etc.).

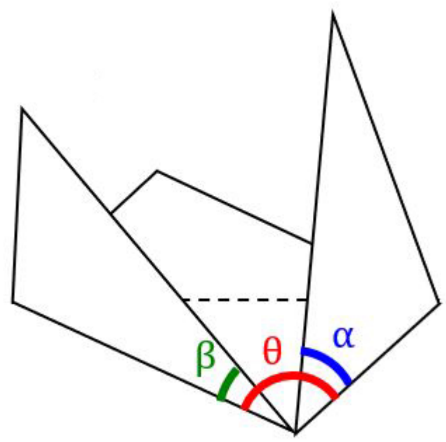



Figure 3

The following task is a scaffold towards further investigations of the tetrahedrons thus formed. It encourages the student to create a net of a solid – a task that demands imagination, spatial understanding and reasoning.

**Task 3:** Consider any of the three types of acute isosceles ΔABC with AB = AC. (Note that this includes equilateral as a special case.) Let D, E and F be the midpoints of the sides BC, CA and AB respectively. We know that ΔAEF, ΔBFD, ΔCDE and ΔDEF will be the four faces of a tetrahedron (see Figure 4). Now to visualize the height of this solid, it is good to split it in two halves. Note that the plane of symmetry of this tetrahedron passes through the line of symmetry of ΔABC. So, divide ΔABC along its line of symmetry AD (which intersects EF at P) and cut along AD. Fold ΔADC along the previous fold-lines to get the halved tetrahedron. Observe that this is a hallow tetrahedron with three faces viz. ΔCDE, ΔAEP (= ½ of ΔAEF) and ΔDEP (= ½ of ΔDEF). What are the sides of the missing face? Construct this missing face (on a separate piece of paper) and attach it to the net ΔADC along PD. Let Q be the third vertex of this missing triangular face. Make sure this triangle is oriented correctly. The side adjacent to AP should coincide with AP when folded. Similarly, the one next to DC should match DC. Fold this (pentagonal) net QPACD to form the halved tetrahedron and check that the fourth face fits properly.

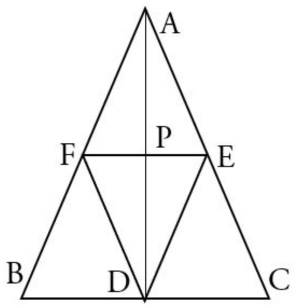



Figure 4

**Teacher Note:** The sides of the fourth triangle must match those of the remaining three faces. Since AE folds up with CE to form one edge of the new tetrahedron, the edges of the fourth face will be equal to AP, PD and DC. So, the fourth face is a triangle ΔDPQ on DP such that DQ = DC and PQ = AP i.e. A, C and Q will coincide in the new tetrahedron (Figure 5).

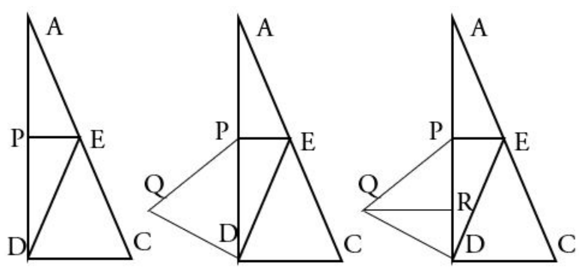



Figure 5

Note that with the fourth face, we can also draw the height of the tetrahedron which is the same as the height QR from Q to PD in ΔDPQ.

It is advisable that the net shown in Figure 5 be made for various measures of ∠ACB while the side length BC remain constant (say 12cm). Different nets can be made for the following values of ∠ABC – 50°, 55°, 60° and 70° (more can be made,

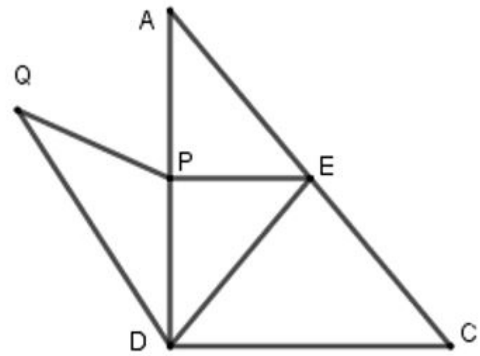



Figure 6

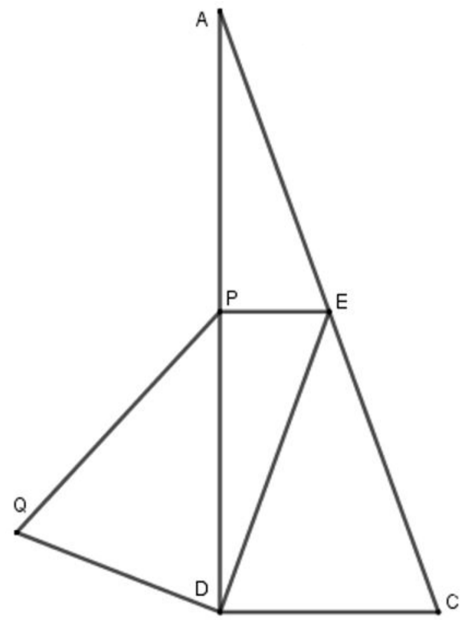



Figure 7

but these are crucial; Figures 6 and 7 include the nets corresponding to 50° and 70° respectively).

**Task 4:** ΔDPQ is an isosceles triangle since PD = PQ.

a. How does ∠DPQ vary with ∠ACB? Do we get all possible isosceles types described in Task 1?

b. What is the side ratio AC : BC for ΔDPQ to be a right triangle?

c. What is the ratio for ΔDPQ to be equilateral?

**Teacher Note:** Different groups of students can be given different values of ∠ACB and asked to create the net shown in Figure 5. Their nets can be then compared to gain further insights.

$\angle ACB$ can vary from 45° to 90° (both excluded). As $\angle ACB$ increases, the height AD of $\Delta ABC$ increases. So, for $\Delta DPQ$, DQ remains fixed, but PD = PQ = ½ AD increases as $\angle ACB$ increases. Therefore, $\angle DPQ$ decreases as PD increases i.e. $\angle ACB$ increases. $\angle DPQ$ is obtuse for $\angle ACB$ = 50° and acute for $\angle ACB$ = 70°. A GeoGebra exploration with a slider for $\angle ACB$ nicely demonstrates how $\angle DPQ$ varies with the former.

**GeoGebra steps**

Chose B = (–6,0) and C= (6,0), and a slider for $\theta$ from 45° to 90°

*D:* midpoint of BC

Rotate B clockwise about C by $\theta$ to get B'

*A:* intersection of the line B'C and the y-axis

*E:* midpoint of AD, P: midpoint of AD

Draw circles (i) centred at P through A, and (ii) centred at D through C

*Q:* intersection of these two circles

Join line segments AD, AC, DC, PE, DQ and PQ

If $\Delta DPQ$ is right angled, then PD : DQ = $1 : \sqrt{2}$. Let us take DQ = DC = ½BC = $2a$. So, PD = $\sqrt{2}a$. Also, PE = ½ × DC = $a$. So, CE = DE = $\sqrt{(PD^2 + PE^2)} = \sqrt{3}a$. Therefore AC = 2CE = $2\sqrt{3}a$ and BC = 2DC = $4a$ i.e. AC : BC = $\sqrt{3} : 2$.

If $\Delta DPQ$ is equilateral, then PD = DQ = $2a$. So, DE = $\sqrt{5}a$ and therefore AC : BC = $\sqrt{5} : 2$

Now that we know the different possibilities for $\Delta DPQ$, it is a good idea to make the nets of the halved tetrahedron (as shown in Figure 5) for the following ratios of AC : BC – (i) 3 : 4, (ii) $\sqrt{3} : 2$, (iii) 1 : 1, (iv) $\sqrt{5} : 2$ and (v) 3 : 2. One option is to keep BC = 12cm and use AC = 9cm, $6\sqrt{3}$cm, 12cm, $6\sqrt{5}$cm and 18cm respectively. These should generate (i) obtuse, (ii) right, (iii) acute type (b), (ii) equilateral and (v) acute type (a) $\Delta DPQ$ respectively.

Different groups of students can tackle the different nets corresponding to the five types of $\Delta DPQ$ from here on.

From here onwards, R is the foot of the perpendicular from Q (where A, B and C coincide) to $\Delta DEF$. By symmetry, this perpendicular or 'height' lies on the fourth face of the halved tetrahedron i.e. $\Delta DPQ$, and R lies on PD where P is the midpoint of EF as indicated in Figure 4 and Figure 5.

The remaining tasks deal with the position of R on the mid-line PD of the base. The aim is to find R's position w.r.t. the sides of the tetrahedron and explore when R coincides with the special points viz. the centroid, the circumcentre, the incentre and the orthocentre– of $\Delta DEF$.

**Task 5:** Since $\Delta DPQ$ can be any of the five possible isosceles triangles, how does the foot of the perpendicular R vary?

a. Is it always inside PD?

b. If not, when is it outside? What does that mean for the tetrahedron?

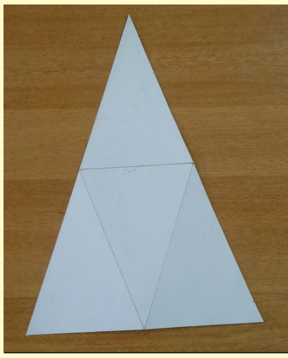
c. What is the border line case? What does that mean for the tetrahedron?

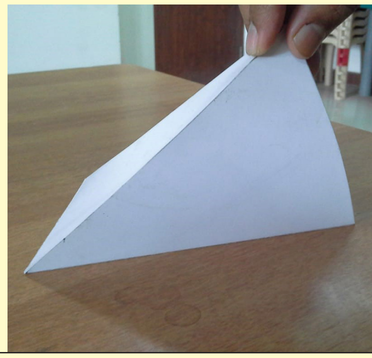
**Teacher Note:** This may come as a surprise but is actually a natural consequence of the types of $\Delta DPQ$. The foot of the perpendicular R is inside PD (and therefore inside the base $\Delta DEF$) iff $\angle DPQ$ is acute (Figure 8).

When $\angle DPQ$ is obtuse, R is outside PD. So, the foot of the perpendicular is outside the base $\Delta DEF$. In this case, R would be on the ray DP, such that DR > DP. The edge AP (= QP) would lean outward from the base $\Delta DEF$ (Figure 9).

The border line case is when $\angle DPQ$ is a right angle. Then R and P coincide, and QP is perpendicular to the base (Figure 10). Table 2 includes the nets of all five new tetrahedrons with R and QR $\perp$ PD marked in each.

Acute isosceles type (a) triangle → folded to tetrahedron

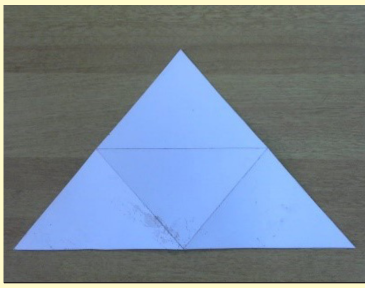


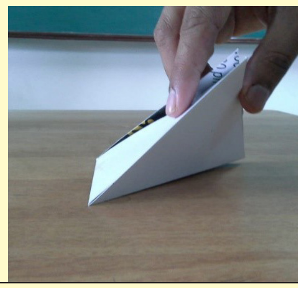


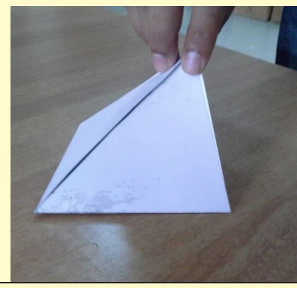
Note that the foot of the perpendicular seems to be inside the base i.e. all 3 faces of the tetrahedron are leaning 'in'.

Figure 8

Acute isosceles type (b) triangle → folded to tetrahedron



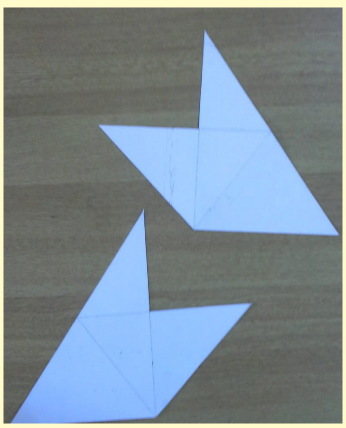


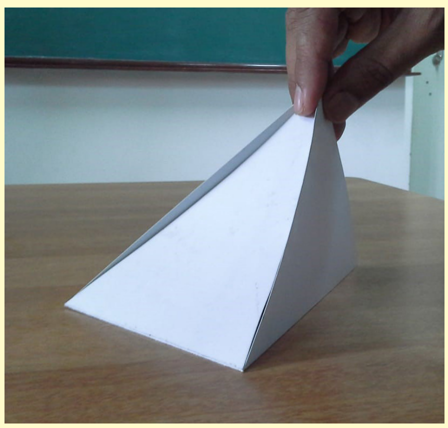


Note that the foot of the perpendicular seems to be outside the base i.e. one face is leaning 'out'.

Figure 9

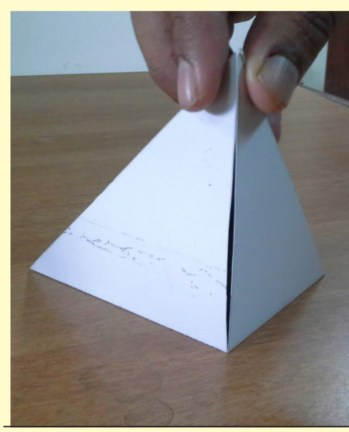
Nets of halved tetrahedron → folded to tetrahedrons (together)





Note that the foot of the perpendicular seems to be on an edge of the tetrahedron i.e. one face (the one that is halved) is perpendicular to the base.

One of the halved tetrahedrons



Another view – this is the 4$^{th}$ face $\Delta DPQ$

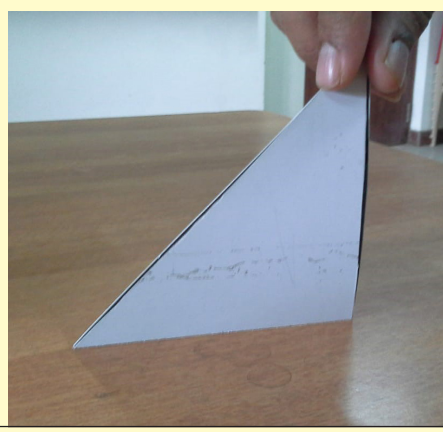


Figure 10

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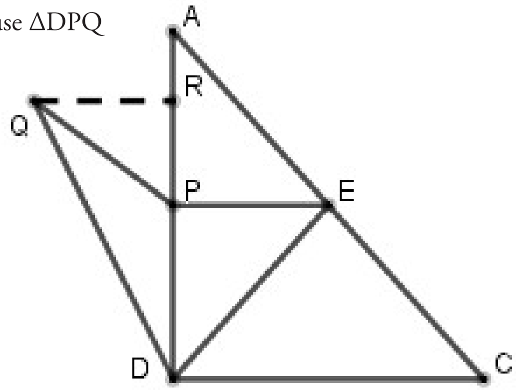
<td>AC : BC = 3 : 4<br>Obtuse ΔDPQ<br>
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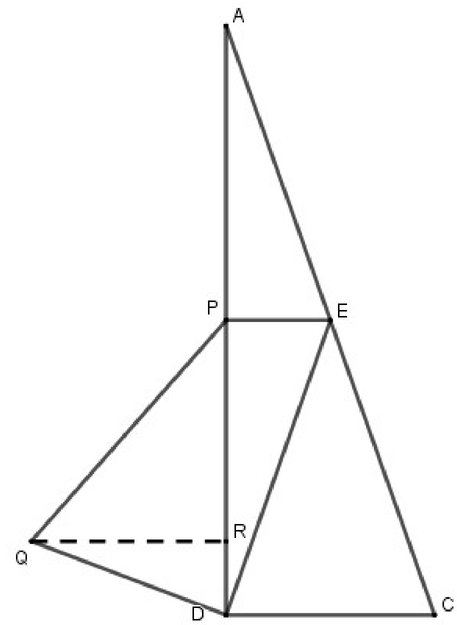
<td rowspan="2">AC : BC = 3 : 2<br>Acute type (b) ΔDPQ<br>
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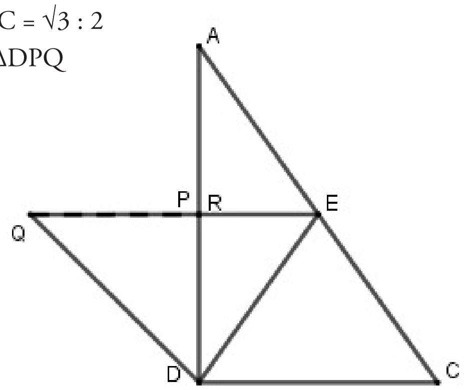
<td>AC : BC = √3 : 2<br>Right ΔDPQ<br>
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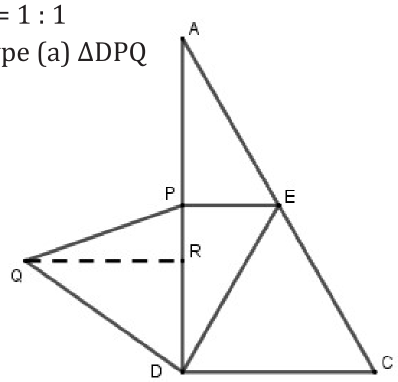
<td>AC : BC = 1 : 1<br>Acute type (a) ΔDPQ<br>
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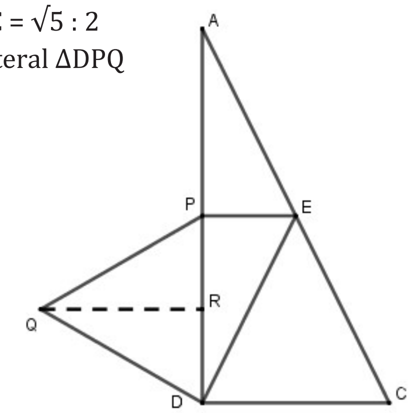
<td>AC : BC = √5 : 2<br>Equilateral ΔDPQ<br>
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Table 2

Having established the variance of R along the ray DP, the next questions are related to the special points of ΔDEF along PD, viz the centroid, the circumcentre, the incentre and the orthocentre of the base. Since ΔABC and therefore ΔDEF are acute triangles, all four of these points lie inside the base. The next task is about locating these special points on the net.

**Task 6:** Consider the net of the halved tetrahedron as shown in Figure 5. PD is the line of symmetry for the base ΔDEF. So, all four of the special points lie on PD. Locate each of these points on PD viz.

a. The centroid G of ΔDEF

b. The incentre I of ΔDEF

c. The circumcentre S of ΔDEF

d. The orthocentre O of ΔDEF

**Teacher Note:** The challenge is to find these points on the net that has only half of ΔDEF. So, properties of these points and the symmetry of isosceles triangle are to be utilized. One needs to draw the net on larger paper and not cut it out, so that the necessary constructions can be done.

The centroid G is a point on the median PD such that PG = 1/3 × PD. So, PD has to be trisected to find G. There is an alternative way: complete ΔDEF (such that P is the midpoint of EF) and construct one more median.

The incentre I lies on the angle bisectors. So, construct the bisector of ∠DEP and let it intersect PD at I.

Similarly, the circumcentre lies on the perpendicular bisector of each side. PD is the perpendicular bisector of EF. So, construct the perpendicular bisector of DE and let it intersect PD at S.

The orthocentre is a bit tricky. Extend EP to F so that FP = PE. Drop perpendicular from F to DE. Let it intersect PD at O.

All of these can be done on GeoGebra as well.

**GeoGebra steps**

(continued from before)

*F:* midpoint of AB

*H:* midpoint of DE, join FH

**Centroid:** G: intersection of FH and AD

*b:* angle bisector of ∠PED

**In-centre:** I: intersection of AD and b

*c:* perpendicular bisector of DE

**Circumcentre:** S: intersection of AD and c

*d:* perpendicular from F to DE

**Orthocentre:** O: intersection of AD and d

An interesting question at this point would be to explore if R coincides with any of these special points. In particular, are there different AC : BC ratios for each of these points? It can be worked out by computing various lengths and doing some tedious algebraic crunching for each of G, S, I and O. However, an alternative approach with the nets provides deeper understanding of the situation.

**Task 7:** Mark G, S, I, O and R on PD on all the nets corresponding to the five types of ΔDPQ. What do you observe?

**Teacher Note:** Considering all five nets and the five points marked in each of them, the following emerge:

1. All the five points coincide for the regular tetrahedron (as expected)
2. The points are always in the same sequence R-S-G-I-O
3. Including P and D, the order is D-R-S-G-I-O-P, for ∠ACB > 60° (Figure 11) and reverses to P-R-S-G-I-O-D for ∠ACB < 60° (Figure 12) – the pink cross indicates the position of A for regular tetrahedron i.e. ∠ACB = 60°

While 1 is quite obvious, it can be rigorously proved. The math hungry can be engaged with the task. The reverse question can also be posed to them i.e. finding AC : BC when R coincides with (i) G, (ii) S, (iii) I and (iv) O. Circumradius and inradius can be computed in terms of AB and BC.

Observation 2 is mostly known except for R. Those who are further interested can consider the various ratios along the line segment SO. Students may be introduced to the nine-point circle after this.

But in one step, this makes it clear that R coincides with all these points only for the regular tetrahedron. For any other tetrahedron or in other words if ∠ACB ≠ 60°, R remains outside the line segment SO.

These explorations start with something as basic as types of triangles which are 2D but soon leap into 3D. There it demands imagining a particular solid and unfolding the same to generate its net, and thus bringing it back to 2D. But the fun starts after that when multiple nets are created by varying an angle (or a side).

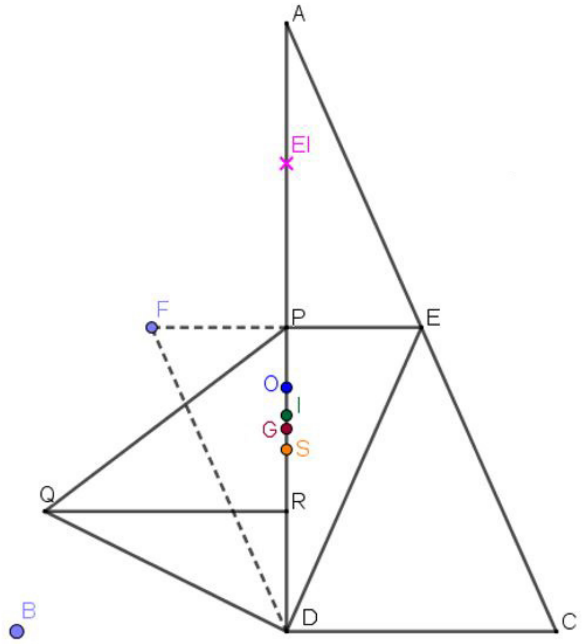

Figure 11

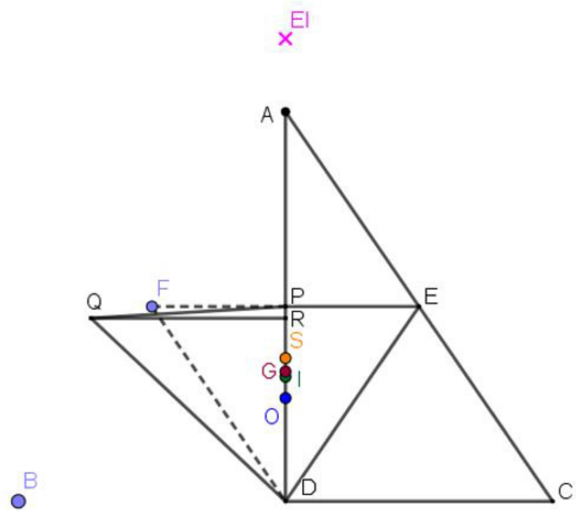

Figure 12

This angle (or side or the ratio of the sides) can be considered as an independent variable. Students get a glimpse of how other angles and position of some points (or some lengths) vary with these independent variables. This raises new questions especially about critical points when one parameter (e.g. angle or length) changes while some remain fixed. It also provides insight into the range of variation and the various possibilities.

We would like to leave the reader with a last question: Try Task 6 for the special points of ΔABC. You will be pleasantly surprised! We hope to dive into that in a subsequent article.

We would like to thank Dr. Prabuddha Chakraborty, Indian Statistical Institute for triggering the later parts of these explorations with the questions related to the foot of the perpendicular and the centres of the base triangle.

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