Relations Among Lengths and Angles in General Parallelograms

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This article aims to derive some relations between lengths and angles in general parallelograms (GPs), i.e., parallelograms that are neither rectangles nor rhombi. We shall use Figure 1 for reference. In parallelogram $ABCD$, $AB > AD$, and $\measuredangle DAB$ is acute. Certain angles have been named p, q, r, s and t.

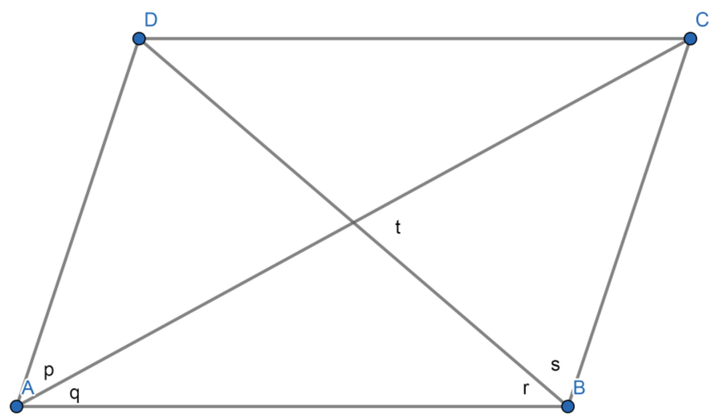

Figure 1.

Once the lengths AB, AD and $\measuredangle A$ are given, a unique parallelogram is defined. The lengths of the diagonals and angles p, q, r, s and t get fixed. (Do you see why?) We shall obtain expressions relating these to the given sides AB, AD and $\measuredangle A$.

Keywords: parallelograms, angles, sides, cosine rule, sine rule

The cosine rule of triangles can be used to obtain the following relations between the sides and the diagonals.

$$AC^2 = AB^2 + AD^2 + 2AB.AD\cos A$$
$$BD^2 = AB^2 + AD^2 - 2AB.AD\cos A.$$

These relations can be reworked to give

$$2(AB^2 + AD^2) = AC^2 + BD^2.$$

This relation is equivalent to Apollonius's theorem. It enables one to obtain one of the lengths given the other three.

The above pair of equations also lead to

$$\cos\measuredangle DAB = (AC^2 - BD^2)/4AB.AD,$$

enabling one to obtain $\measuredangle DAB$ from the four lengths.

Using the sine rule in triangles ACD, ABC, ABD and BCD in turn, we obtain the following relations between angles p, q, r, s and $\measuredangle DAB$. Note that $\sin D = \sin(180 - A) = \sin A$.

$$\sin p/\sin\measuredangle DAB = \sin p/\sin\measuredangle ADC = \sin\measuredangle CAD/\sin\measuredangle ADC = CD/AC = AB/AC,$$
$$\sin q/\sin\measuredangle DAB = \sin q/sin\measuredangle CBA = \sin\measuredangle CAB/\sin\measuredangle ABC = BC/AC = AD/AC,$$
$$\sin r/\sin\measuredangle DAB = \sin\measuredangle ABD/\sin\measuredangle BAD = AD/BD,$$
$$\sin s/\sin\measuredangle DAB = \sin s/\sin C = \sin\measuredangle CBD/\sin\measuredangle BCD = CD/BD = AB/BD.$$

To obtain a relation between angle t and $\measuredangle A$, we note the following.

Area of parallelogram $ABCD = AB.AD\sin A = \frac{1}{2}AC.BD\sin t$, giving

$$\sin t/\sin A = 2AB.AD/AC.BD$$

To summarise

$$\sin p/\sin A = AB/AC$$
$$\sin q/\sin A = AD/AC$$
$$\sin r/\sin A = AD/BD$$
$$\sin s/\sin A = AB/BD$$
$$\sin t/\sin A = 2AB.AD/AC.BD$$

Observe the neat pattern in the above set of equations. Several inferences can be made from the above.

(i) $\sin p \cdot \sin r = \sin q \cdot \sin s$

(ii) Angle inequalities: $q < p < s$ since $\frac{AD}{AC} < \frac{AB}{AC} < \frac{AB}{BD}$, and $q < r < s$ since $\frac{AD}{AC} < \frac{AD}{BD} < \frac{AB}{BD}$.

(iii) In the general case, angles p, q, r, s are distinct. Can any two of these be equal in a General Parallelogram (GP)?

If $p = q$, then $\measuredangle DAC = \measuredangle ACD$. Consequently $AD = DC$ and the figure becomes a rhombus. A similar situation arises if $r = s$. If $q = r$ or $p = s$, then the diagonals are of the same length, making the figure a rectangle. If $q = s$, then by the above inequalities, $p = q = r = s = 45°$ and the figure turns out to be a square.

We now examine the pair p, r. If $p = r$, then we have $AB/AC = AD/BD$ or $AB/AD = AC/BD$. That is, the ratio of the (adjacent) sides equals the ratio of the diagonals. This is true of a square. We ask if this is possible in a GP.

Taking $AB = 1$, $AD = x < 1$, $AC = y$ and $BD = xy$, we have

$2(1 + x^2) = y^2 + x^2y^2 = y^2(1 + x^2)$, leading to

$y = \sqrt{2}$, i.e., the longer diagonal must be $\sqrt{2}$ times the longer side for angles p and r to be equal. The ratio of AD to AB can be any value between $(\sqrt{2} - 1)$ and 1. To see why, consider triangle ABD and apply the triangle inequality to its sides, which are $1, x, x\sqrt{2}$.

If $p = r$ then $t = \measuredangle A$. (This follows from the relations $\measuredangle A = p + q$ and $t = q + r$.) That is, the angle between the sides equals the angle between the diagonals. This is again true of a square but is also true of the kind of GP described above.

DUAL FIGURE CORRESPONDING TO A PARALLELOGRAM

Every parallelogram has its dual figure, whose sides are parallel to the diagonals of the former and vice versa. (Note that the concern here is only the *shape* of the figure, and not the scale. We can scale up the dual figure by any desired scale factor.) An easy way to obtain the dual of any parallelogram is to connect the mid-points of the sides sequentially, as indicated in Figure 2.

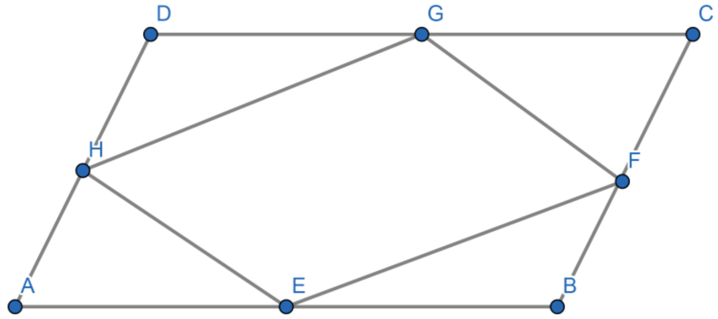

Figure 2

A rectangle and a rhombus where the angle between the sides in one is equal to the angle between the diagonals in the other are duals of each other. The dual of a square is another square. A GP of the type described in the previous paragraph, i.e., where the longer diagonal is $\sqrt{2}$ times the longer side, is its own dual.

To summarise, a member of this class of GPs shares the following properties with a square:

Ratio of sides equals ratio of diagonals. The longer diagonal is $\sqrt{2}$ times the longer side, while the shorter diagonal is $\sqrt{2}$ times the shorter side. The angle between the diagonals equals angle between sides It is its own dual.

We shall call a GP of this type a 'PSEUDOSQUARE'.

Editor's note. In the March 2017 issue of AtRiA, author Michael de Villiers describes a 'Golden rectangle' (pp. 64-69). In this figure the (adjacent) sides as well as the diagonals are in the ratio $\varphi : 1$, φ being the Golden ratio. Interestingly, the acute angle of the figure turns out to be 60°. Such a golden rectangle is a special case of a pseudosquare, as defined above, where the longer diagonal is $\sqrt{2}$ times the longer side. It can be shown that if the acute angle of a pseudosquare is 60°, then the sides and diagonals are in the golden ratio, and vice versa.

A COORDINATE GEOMETRY BASED APPROACH TO PSEUDO-SQUARES

With reference to Figure 3, $ABCD$ is a parallelogram with $AB = 1$ unit; A is placed at the origin and B on the positive X-axis. Angle q is an arbitrary angle less than $45°$. $AC = \sqrt{2}$ units. Point D completes the parallelogram. The coordinates of A, B, C, D are as indicated in the figure.

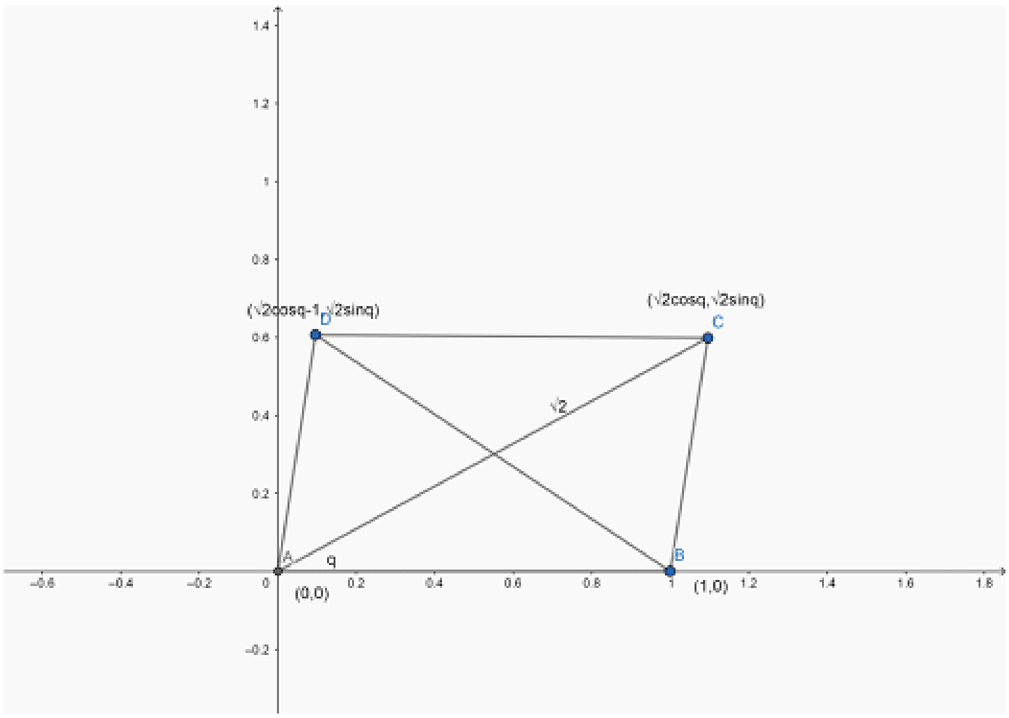

Figure 3

We first show that the ratio of sides equals the ratio of diagonals.

$AB = 1$. $AC = \sqrt{2}$.

$AD^2 = (\sqrt{2}\cos q - 1)^2 + (\sqrt{2}\sin q)^2$, which simplifies to $3 - 2\sqrt{2}\cos q$, hence

$AD = \sqrt{[3 - 2\sqrt{2}\cos q]}$.

$BD^2 = (\sqrt{2}\cos q - 2)^2 + (\sqrt{2}\sin q)^2$, which simplifies to $2(3 - 2\sqrt{2}\cos q)$, hence

$BD = \sqrt{[2(3 - 2\sqrt{2}\cos q)]}$.

Hence $BD/AD = \sqrt{2} = AC/AB$.

Next, we show that the angle between the sides equals the angle between the diagonals.

Slope of $AD = \frac{\sqrt{2}\sin q}{\sqrt{2}\cos q - 1}$, while

Slope of $AB = 0$.

tan value of the angle between the lines is $\frac{\sqrt{2}\sin q}{\sqrt{2}\cos q - 1} = \frac{2\sin q}{2\cos q - \sqrt{2}}$

Slope of $BD = \frac{\sqrt{2}\sin q}{\sqrt{2}\cos q - 2}$, while

slope of $AC = \frac{\sqrt{2}\sin q}{\sqrt{2}\cos q}$.

tan of the angle between the lines is again $\frac{2\sin q}{2\cos q - \sqrt{2}}$

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