A geometric method to locate Neptune

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We develop a direct geometric method to determine the orbital parameters and mass of a planet, and we then apply the method to Neptune using high-precision data for the other planets in the solar system. The method is direct in the sense that it does not involve curve fitting. This paper, thereby, offers a new pedagogical approach to orbital mechanics that could be valuable in a physics classroom.

I. INTRODUCTION

The discovery of Neptune in 1846 by observing deviations of Uranus from its predicted orbit remains one of the crowning achievements of Newtonian mechanics.^{1–3} The discoverers faced a difficult inverse problem, easily solvable today with electronic computers, but Herculean in the mid-nineteenth century.^{4,5} The discoverers, Urbain Le Verrier and John Couch Adams, independently investigated Alexis Bouvard's ephemeris tables for Uranus to check their accuracy and found no computational errors.^{4–6} With this, it became clear that the problem lay not with the tables, but rather with the planet itself! The mathematical problem of locating the precise position and determining the mass of the perturbing object was non-trivial;^{5,6} Le Verrier and Adams proceeded by linearizing the system of equations involved and solving them by least squares. Computing the semi-major axis of the perturbing body was also a complex problem, for which an informed estimate had to be made. This was done using the Titius-Bode law,⁷ which held for the five planets known since antiquity. Neptune has subsequently been "rediscovered" with simpler methods several times after its real discovery.^{6,8–12}

In today's age of high-precision data, orbital state vector data of most solar system bodies are available.¹³ We have retrieved these data for the major planets in *Cartesian coordinates*, with the origin at the instantaneous center of the Sun (see the Appendix for more details). These data, collected at a time step of 2 h, range from the date of Uranus' discovery (March 1781) to the present day (March 2020). It is clear that we have an advantage over the discoverers, in terms of the nature, quality and duration of data at hand. Using this advantage, we describe a purely geometric way of locating a planet, focusing on Neptune.

Since we work with state vector data and not the orbital parameters of the known planets, our approach naturally lends itself to vector analysis. The geometrical ideas are encoded into scalar (dot) products and vector (cross) products. The method emphasizes basic ideas of classical mechanics, such as Newton's laws, Kepler's laws, and motion in the presence of an inverse square law force, but does not use curve fitting to locate the unknown planet. All these ideas are typically encountered in an undergraduate physics curriculum, making our method better suited to pedagogical purposes than the historical ones.

II. EQUATION OF MOTION

Our model comprises *only* nine objects: the Sun, the seven known planets at the time (Mercury to Uranus) and Neptune, which we consider unknown for the purpose of demonstrating our method.¹⁴ Newton's law written for

the ith object reads

$$\frac{d^2 x_i}{dt^2} = -\sum_{j \neq i} GM_j \frac{x_i - x_j}{|x_i - x_j|^3} \qquad (i = 1 \text{ to } 9), \quad (1)$$

where G is the universal gravitational constant, \boldsymbol{x}_i s and \boldsymbol{x}_j s stand for the position of the center of mass (COM) of the objects (including their moons, if any), and M_j s are the masses (again including the moons, if any). However, we need not worry about solving these coupled differential equations, for we have the solution in terms of data for all objects except Neptune, whose parameters need to be determined using these equations.

The $d^2 \boldsymbol{x}_i/dt^2$ in Eq. (1) denote accelerations with respect to an inertial frame. The COM of the solar system is a reasonably good choice for such a frame. However, the COM is not an observable, but rather a construct that can only be obtained after having knowledge of *all* constituents of the system, including Neptune (N), which is yet to be "discovered." We circumvent this problem by subtracting Eq. (1) written for the Sun (\odot) from the same equation written for Uranus (U). This gives

$$GM_N\left(\frac{\boldsymbol{r}_N - \boldsymbol{r}_U}{|\boldsymbol{r}_N - \boldsymbol{r}_U|^3} - \frac{\boldsymbol{r}_N}{|\boldsymbol{r}_N|^3}\right) = \boldsymbol{V}(t), \qquad (2)$$

where

$$\mathbf{V}(t) = \frac{d^2 \mathbf{r}_U}{dt^2} + G\left(M_{\odot} + M_U\right) \frac{\mathbf{r}_U}{|\mathbf{r}_U|^3} - \sum_{j \neq \odot, U, N} GM_j \left(\frac{\mathbf{r}_j - \mathbf{r}_U}{|\mathbf{r}_j - \mathbf{r}_U|^3} - \frac{\mathbf{r}_j}{|\mathbf{r}_j|^3}\right) \quad (3)$$

and where $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_{\odot}$, which are relative coordinates with respect to the Sun. It is instructive to write down all the terms explicitly to understand the derivation of Eqs. 2 and 3, and to note that $\mathbf{r}_i - \mathbf{r}_j = \mathbf{x}_i - \mathbf{x}_j$.

In Eq. (3), the \mathbf{r}_i s are observable without a priori knowledge of Neptune. Thus, with Eq. (2) and the assembled data from Ref. 13 (as elaborated in Sec. I), $\mathbf{V}(t)$ is known; see Sec. III for a physical understanding of this vector. The stage is now set to find the unknowns $\mathbf{r}_N(t)$ and M_N , which we do through a purely geometric approach.



FIG. 1. The behavior of \hat{V} as Neptune (blue sphere—the one on the outer orbit) moves around in its orbit, with Uranus (brown sphere—the one on the inner orbit) held fixed. The Sun (not shown) is at the center and the black dashed line is the z-axis. The orbits of Uranus (brown inner orbit) and Neptune (blue outer orbit) are both shown to be slightly inclined to the x-y plane. The red solid line denotes \hat{V} . The green dot-dashed line denotes vector p (Eq. (4)). (a) and (c) correspond to conjunction and opposition of the Sun–Uranus– Neptune system, respectively. (b) and (d) correspond to configurations, which are not conjunction or opposition, but still satisfy $\chi = 0$ (see Eq. (5) and the discussion around it). Note that this figure is a schematic for the sake of visualizing \hat{V} and was not derived from any data.

III. UNDERSTANDING THE VECTOR V

Since our discussion revolves around the vector V, it is important to discuss what it means physically. From Eq. (2), V(t) would be zero if Neptune did not exist; it is that part of the acceleration of Uranus not explained by the other planets.

Consider \hat{V} , which is the unit vector along V. In Fig. 1, this is denoted by a red solid vector. If r_U is imagined to be held fixed and r_N is made to rotate anticlockwise once around the z axis, then, simultaneously, \hat{V} will complete two anticlockwise circuits around the z axis. We will revisit \hat{V} in Sec. IV.

We now turn our attention to $V(t) = |\mathbf{V}(t)|$, the magnitude of \mathbf{V} . This is shown in Fig. 2.¹⁵ With \mathbf{V} being



FIG. 2. V, which is the magnitude of V (Eq. (3)), is shown as a function of time. The two peaks correspond to November 1822 and June 1994. As argued in Sec. III, these peaks provide a rough estimation of the time of conjunction of the Sun– Uranus–Neptune system.

that part of Uranus' acceleration arising exclusively from Neptune, the peaks in Fig. 2 are expected to correspond to conjunctions. This can be confirmed by modeling the orbits of Uranus and Neptune as co-planar circles around the Sun, and obtaining from Eq. (2) that V peaks during conjunction, irrespective of the radii of the orbits. Hence, it is reasonable to expect that the peaks in Fig. 2, which occurred in November 1822 and June 1994, signify conjunctions. However, this prediction is not precise as the planets' orbits are neither exact circles nor exactly co-planar. Therefore, the analysis, thus far, provides but a rough estimate for the date of conjunction.

IV. DETERMINATION OF CONJUNCTION AND OPPOSITION WITH URANUS

To obtain precise conjunction and opposition times, we recall that in a spherical polar coordinate system with the Sun at the origin, the azimuthal angle (ϕ) for both Uranus and Neptune will be the same during conjunction, and differ by π radians at opposition. From Eq. (2) and Fig. 1, it is clear that during *both* conjunctions *and* oppositions, the vector V will have the same ϕ coordinate as Uranus. This observation motivates us to define a vector

$$\boldsymbol{p}(t) = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}_U(t), \tag{4}$$



FIG. 3. Precise determination of the conjunction and opposition of the Sun–Uranus–Neptune system. χ (Eq. (5)) is shown as a function of time. Among the several roots of χ in the duration shown, the ones occurring in August 1821 and March 1993 determine conjunctions (C_1 and C_2), while the one in May 1908 determines opposition (O_1). The origin of the remaining roots (also depicted in parts (b) and (d) of Fig. 1) is discussed below Eq. (5).

which lies in the x-y plane and is normal to $\mathbf{r}_U(t)$ (denoted as a green dot-dashed line in Fig. 1). In terms of \mathbf{p} , we can hope to determine a conjunction or an opposition by checking whether \mathbf{V} is perpendicular to \mathbf{p} at a given time. In other words, we define a quantity χ , which is a measure of the projection of $\hat{\mathbf{V}}$ along \mathbf{p} , as

$$\chi(t) = \hat{\boldsymbol{V}}(t) \cdot \boldsymbol{p}(t), \qquad (5)$$

and look for its roots.

Figure 3 shows χ as a function of time. As expected, χ does vanish around the estimated times for conjunctions and oppositions as obtained in the Sec. III by looking at the peaks in Fig. 2. We have already seen that the ϕ coordinate of V is the same as that of Uranus during both conjunctions and oppositions. This observation can also be used to further verify whether a given root of χ corresponds to either a conjunction or an opposition.

The remaining roots of χ , which do not correspond to conjunctions or oppositions can be understood as follows. During conjunctions and oppositions, the projections of both the bracketed terms in Eq. (2) on p vanish. However, χ can also vanish if the projections of these two terms cancel each other. The relative dominance of these terms flips when the system moves from a conjunction to



FIG. 4. The projection of V(t) along the normal to the orbital plane of Uranus, quantified through ξ (Eq. (6)), is shown as a function of time. The roots of ξ in the duration shown occur at June 1791, September 1850, June 1860, March 1932, April 1962, and May 2015. As argued in Sec. V, the roots in September 1850 and May 2015 can be used to precisely determine the orbital period of Neptune.

an opposition or vice versa, resulting in an extra root between them. Basically, if the *n*th root corresponds to a conjunction, the (n+2)th root will be an opposition, the (n+4)th root will again be a conjunction, and so on.

In this manner, we conclude that the roots of χ in Fig. 3, which happen to fall in August 1821 and March 1993, are conjunctions and designate them as C_1 and C_2 , respectively. The root falling in May 1908 then corresponds to an opposition, which we call O_1 .

V. DETERMINATION OF THE ORBITAL PERIOD AND SEMI-MAJOR AXIS

The synodic period of Neptune with respect to Uranus is the time difference between the conjunctions C_1 and C_2 : $T_{sy} = 171.6$ years. If Uranus and Neptune were indeed co-planar and moved at constant angular speeds, this information, along with the orbital period of Uranus, would be sufficient to calculate the sidereal period of Neptune around the Sun to be $T_N \approx 165$ years. However, this is an approximation.

We, therefore, venture to obtain T_N precisely by defining

$$\xi(t) = \hat{\boldsymbol{V}}(t) \cdot \hat{\boldsymbol{n}}_U, \qquad (6)$$

where $\hat{\boldsymbol{n}}_U$ is a unit vector along $(\boldsymbol{r}_U imes rac{d\boldsymbol{r}_U}{dt})$ evaluated at

our model's epoch, March 1781 (referred to hereafter as T_I). In other words, $\hat{\boldsymbol{n}}_U$ is a unit normal to the orbital plane of Uranus. The quantity ξ , a measure of projection of V(t) along \hat{n}_U , is shown in Fig. 4 as a function of time. When Neptune crosses the orbital plane of Uranus, ξ vanishes. However, similar to the roots of χ , between any two consecutive roots of this kind will exist another kind of root, where the components along \hat{n}_U of the two terms bracketed in Eq. (2) cancel each other. Thus, with reference to Fig. 4, if the time difference between the nth and (n + 4)th roots of ξ corresponds to T_N , then that between the (n+1)th and (n+5)th roots corresponds to T_{sy} , and vice versa. In particular, the roots in June 1791 and April 1962 differ by 170.8 years, which is very close to T_{sy} . Hence, the roots in September 1850 and May 2015 can be used to give $T_N = 164.7$ years. The semi-major axis (a) is then obtained from T_N using Kepler's third law (assuming $M_N \ll M_{\odot}$) to be 30.05 astronomical units (AU).

VI. DETERMINATION OF OTHER ORBITAL PARAMETERS AND MASS

To begin here, the polar angle θ and the azimuthal angle ϕ (the spherical polar coordinates) of Neptune during a conjunction with Uranus can be determined as follows.

Equation 2 can be re-expressed as

$$\boldsymbol{r}_{N} = \boldsymbol{r}_{U} + |\boldsymbol{r}_{N} - \boldsymbol{r}_{U}|^{3} \left(\frac{\boldsymbol{r}_{N}}{|\boldsymbol{r}_{N}|^{3}} + \left| \frac{\boldsymbol{r}_{N} - \boldsymbol{r}_{U}}{|\boldsymbol{r}_{N} - \boldsymbol{r}_{U}|^{3}} - \frac{\boldsymbol{r}_{N}}{|\boldsymbol{r}_{N}|^{3}} \right| \hat{\boldsymbol{V}} \right).$$
(7)

As an initial guess, we can take θ for Neptune to be $\pi/2$ radians, and, since we are dealing with a conjunction, ϕ of Neptune can be taken equal to that of Uranus. By freezing Neptune's r coordinate to its semi-major axis a, which has already been obtained, we have a guess for \mathbf{r}_N . With this, every quantity on the RHS of Eq. (7) is known. Hence, the LHS of this equation can be taken as a better estimate for \mathbf{r}_N . This way, we can iterate Eq. (7) to obtain ϕ and θ of Neptune at conjunction.

A similar procedure can be carried out for an opposition by re-expressing Eq. (2) as

TABLE I. The converged directions (as elaborated in Sec. VI) of conjunctions (Uranus and Neptune aligned on the same side of the Sun) and opposition (Uranus and Neptune aligned on the opposite sides of the Sun). ϕ and θ are the spherical polar coordinates, while ψ is the angle in the orbital plane from C_1 .

	Date (year CE)	ϕ (rad)	θ (rad)	ψ (rad)
C_1	August 1821	4.532	1.512	0
O_1	May 1908	1.582	1.647	3.334
C_2	March 1993	4.787	1.490	0.256

$$\boldsymbol{r}_{N} = |\boldsymbol{r}_{N}|^{3} \left(\frac{\boldsymbol{r}_{N} - \boldsymbol{r}_{U}}{|\boldsymbol{r}_{N} - \boldsymbol{r}_{U}|^{3}} - \left| \frac{\boldsymbol{r}_{N} - \boldsymbol{r}_{U}}{|\boldsymbol{r}_{N} - \boldsymbol{r}_{U}|^{3}} - \frac{\boldsymbol{r}_{N}}{|\boldsymbol{r}_{N}|^{3}} \right| \hat{\boldsymbol{V}} \right)$$
(8)

and taking ϕ for Neptune to be π radians away from that of Uranus.

Note that we have re-expressed Eq. (2) in two different ways: one for conjunction (Eq. (7)) and the other for opposition (Eq. (8)). These choices ensure convergence of the iteration algorithm in the respective cases. The converged values of ϕ and θ of Neptune after iteration are given in Table I. Thus, we obtain the directions of Neptune during C_1 , O_1 , and C_2 .

At this stage, we assume that the orbit of Neptune is an ellipse with the Sun at one focus (Kepler's first law). Since the time difference between C_1 and C_2 is greater than the orbital period, it is clear that C_2 is ahead of C_1 in the orbit. The orbital plane is characterized by the inclination (i) of its normal from the z axis and the longitude of the ascending node (Ω) . These can be evaluated from the knowledge of the directions of C_1 and C_2 as i = 0.1124 and $\Omega = 3.9806$, both in radians. The direction of O_1 should be very close to this plane. A negligible component perpendicular to this plane, if any, can be dealt with by projecting O_1 onto the plane. Let ψ be the angle measured in the plane of the ellipse from C_1 . ψ for C_2 and O_1 can be easily found, since we already know the corresponding ϕ and θ . These values of ψ are tabulated in Table I.

Using this information, the equation of the ellipse can be written¹⁶ with unknowns being its eccentricity (ϵ) and the distance of Neptune from the Sun (r_I) at T_I . These



FIG. 5. Determination of eccentricity (ϵ) and the distance of Neptune from the Sun (r_I) at our model's epoch (T_I). The solid line represents the combination of ϵ and r_I that is consistent with ψ during O_1 . The dashed line is the combination of ϵ and r_I that is consistent with ψ during C_2 . Their intersection determines ϵ and r_I of Neptune's orbit.

unknowns can be obtained by demanding that Neptune's orbit assumes the appropriate ψ values given in Table I at the respective times.¹⁷

The combination of ϵ and r_I that are separately consistent with ψ during O_1 and C_2 is obtained as shown in Fig. 5. The intersection of these curves gives the required combination of ϵ and r_I . This gives $\epsilon = 0.0143$ and $r_I = 30.24$ AU. The orbit of Neptune is thus completely determined.

The time at which Neptune passes through the perihelion (T_p) and the argument of the perihelion (angular coordinate of the perihelion in the orbital plane from the ascending node, denoted by ω) emerge as November 1892 and 3.282 radians, respectively.

Thus, we now know *all* the orbital parameters of Neptune, from which its location at any time can be determined.

The only remaining parameter is the mass of Neptune. From Eq. (2), this can be written as

$$GM_N = V / \left| \frac{\boldsymbol{r}_N - \boldsymbol{r}_U}{|\boldsymbol{r}_N - \boldsymbol{r}_U|^3} - \frac{\boldsymbol{r}_N}{|\boldsymbol{r}_N|^3} \right|.$$
(9)

Now that we know \mathbf{r}_N , we can substitute it into Eq. (9) to obtain M_N . The mass should of course be a constant, but any deviation of the position obtained from the exact value will cause a variation in M_N . Hence, we averaged

the M_N values so obtained over the orbital period to arrive at $M_N = 1.033 \times 10^{26}$ kg.

VII. COMPARISON WITH ACTUAL VALUES

If $\mathbf{r}_N(t)$ is the trajectory obtained here and $\mathbf{r}_{N0}(t)$ is the actual trajectory of Neptune given by Ref. [13], one way to characterize the error is

% deviation =
$$\frac{|\boldsymbol{r}_N(t) - \boldsymbol{r}_{N0}(t)|}{|\boldsymbol{r}_{N0}(t)|} \times 100.$$
 (10)

The deviation of $\mathbf{r}_N(t)$ from $\mathbf{r}_{N0}(t)$ varies with time and is always within 1.7%. As viewed from Earth, the predicted direction of Neptune always lies within one degree of its actual direction.

Another approach to testing the quality of our solution is by characterizing the elliptical orbit through its six elements. The inclination (i) of the orbital axis and the longitude of the ascending node (Ω) specify the plane of the orbit. The semi-major axis (a) and eccentricity (ϵ) specify the size and shape of the ellipse. The argument of the perihelion (ω) specifies the orientation of the ellipse in its plane. The time at which Neptune passes through the perihelion (T_p) specifies the epoch.

At any given time, the instantaneous position $(\mathbf{r}_{N0}(t))$ of Neptune as seen from the Sun is already obtained from Ref. [13]. From this, the angular momentum (L) and the energy (E) of Neptune can be calculated. While ϵ and a can be obtained from E and |L|, i and Ω can be obtained from \hat{L} . The calculation of T_p and ω is more involved, and can be determined from the eccentric anomaly.¹⁶ Hence, at each instant, the orbital elements can be calculated. If Neptune and the Sun were isolated, the orbital elements would be constants. However, due to perturbations by other planets, they do vary. We summarize these variations in Table II, which also lists the orbital elements calculated using $\mathbf{r}_{N}(t)$ obtained from our method. It is apparent that they compare well with the corresponding actual values. Note that since $\mathbf{r}_N(t)$ is presumed to be perfectly elliptical, we obtain definite values for the orbital parameters, not a range of values.

VIII. CONCLUSION

The discovery of Neptune was a magnificent example of mathematical and scientific analysis. While the older methods were certainly able to locate Neptune, they were less transparent in terms of the usage of physics principles and geometrical arguments than that advanced here.

Aided with modern-age data, the method we have described to locate Neptune is simple, both conceptually and by means of calculation. Moreover, it is also a *direct* geometric method, without any curve fitting or solving of differential equations. All steps in this method are based on ideas encountered in an undergraduate curriculum, such as vector analysis and the laws of planetary motion. Thus, our method offers a pedagogical alternative to traditional methods.

This method can also be applied to the case of a hitherto undiscovered planet. For this, sound data for a long enough duration (at least until one time period of the planet) of *all* the other planets in the system are required. This method can also serve to provide an initial guess for more sophisticated analyses, be it for application within our solar system or to exoplanetary systems.

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Appendix: DOWNLOADING THE DESIRED DATA

The following settings were made on Ref. 13 to generate and download the data.

(1) Choose "Vectors" from the "Ephemeris Type" menu. This will "generate a Cartesian state vector table of any (solar system) object with respect to any major body."

TABLE II. Comparison of Neptune's orbital parameters and mass, obtained using the method illustrated in this paper, with the actual values of the osculating orbit (as elaborated in Sec. VII) in the time interval considered.

	$i \ (rad)$	Ω (rad)	a (AU)	ε	ω (rad)	T_p (year CE)	M_N (kg)
Our method	0.1124	3.9806	30.05	0.0143	3.282	Nov 1892	1.033×10^{26}
Actual	0.1122 – 0.1123	3.9769 - 3.9778	29.93 - 30.31	0.002 – 0.016	1.918 - 3.633	Aug 1855–Aug 1901	1.025×10^{26}

- (2) Choose the object in question from the "Target Body" menu.
- (3) Write "@sun" in the box provided in the "Coordinate Origin" menu. This will ensure that the coordinate origin is fixed to the center of the Sun at all times.
- (4) From the "Time Span" menu, choose a time span in the appropriate format (specified in the same webpage next to these fields) and the necessary time step.
- (5) Under the "Table Settings" menu, select vector table output as "Type 2 (state vector {x,y,z,vx,vy,vz})." Then, choose the required out-
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- ¹ All dates mentioned in this paper are in CE.
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put units from the "output units" submenu. Now, choose "body mean equator and node of date" from the "reference plane" submenu. This will set the reference x-y plane to coincide with the Sun's mean equator. Then, choose "ICRF/J2000.0" from the "reference system" submenu. Leave "aberrations" as "Geometric states (no aberrations; instantaneous ephemeris states)." Check the "labels," "CSV format," and "object page" options.

(6) From the "Display/Output" menu, check the "download/save" option so that the ephemeris results can be saved to a local file.

Then select the "Generate Ephemeris" option and a .txt file with the ephemeris data will be downloaded.

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- ¹³ J. D. Giorgini and JPL Solar System Dynamics Group, see <https://ssd.jpl.nasa.gov/horizons.cgi> for "NASA/JPL Horizons On-Line Ephemeris System" (last accessed April 04, 2020).
- ¹⁴ Inclusion of more objects such as big asteroids like Ceres will nominally increase accuracy.
- ¹⁵ The short term noise in Figs. 2–4 is removed by taking the moving average.
- ¹⁶ Herbert Goldstein, Charles P. Poole, and John L. Safko, *Classical Mechanics*, 3rd ed. (Pearson Education, 2007), p. 92–103.
 ¹⁷ A. Charles C. Charles P. Poole, and John L. Safko, P. 92–103.
- ¹⁷ Actually, $\zeta = \text{sign of the radial component of velocity at } T_I$ is another unknown parameter while solving for the ellipse. With other parameters fixed, we tried both possibilities ($\zeta = \pm 1$) and have kept $\zeta = +1$ as there is no root analogous to that found in Fig. 5 for $\zeta = -1$.