Features

Why Should You Study Mathematics?

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Introduction and Motivation

My daughter inspired me to author this essay. She will soon begin her life as a university student. I knew this day would come when she would leave home for college. As a parent, and more specifically, an Indian parent, I tried to steer her toward the engineering and computer science fields. Not only do these fields offer the most opportunities but my wife and I believe that our daughter would do very well and contribute to them. Instead, she is choosing to study mathematics.

I admire my daughter for making this choice. Mathematics is one of the oldest subjects and forms the basis for all problem solving. However, mathematics is not the first choice for many incoming college freshmen. It is, I wager, not their second, or even their third choice. In my opinion, there are three common reasons, or myths, behind this.

First, the general belief is that mathematics is a difficult subject and most of us do not have fond memories of our mathematics classes from school. Second, a student of computer science will pursue a career of a computer scientist; a student of aerospace engineering will seek employment as an aerospace engineer. What is the equivalent profession for a graduate with a degree in mathematics? We do not often see employment adverts for a 'mathematician.' Most of us assume that the logical profession for those with a degree in mathematics would be that of a teacher at a school or college, or at the very high end, that of a professor at a premium research institution. Third, compounding this is the hard reality that none of these professions offer the promise of making you a wealthy person as do other professions like IT and medicine.

Keywords: Mathematics, motivation, history, vignettes, applications of mathematics

How can one then be motivated to study mathematics? This article attempts to break these myths and supply an answer to the question an aspiring college student would ask of us: why should I choose a degree in mathematics and a career as a mathematician?

Let me first get this notion out of the way that mathematics is a difficult subject. I am, by profession, a computer scientist. I study algorithms, applications, systems software, and hardware. When I tell people about my profession, most confide in me that they do not understand computers. Many others confess that they have found programming to be difficult. I find biology and medicine, with all those complex and difficult to spell terms, like hydroxychloroquine, to be harrowing. So, difficult is a relative term: one person's difficult is another person's easy. Once a subject is understood, it no longer remains hard or mysterious. To me, all it takes is motivation and perseverance - this applies to any field, not just to mathematics.

So, why *should* you study mathematics?

To Shake the World

Mathematics offers you the opportunity to achieve greatness. One enviable property of this subject is that discoveries in this field are applicable to so many other areas of life. I will provide three examples, two historic, and one relatively recent, of mathematicians whose discoveries have made incredible and lasting impacts on the world.

Newton [1]

Isaac Newton is generally accepted to be one of the most influential human beings of all time. In 1665 he was a relatively unknown undergraduate student at Trinity College in Cambridge when England was hit by the great plague. Newton had to move out of Cambridge and into his grandmother's home in the country. Most of Newton's great discoveries came during the next two years that he lived at his grandmother's. Influenced by the work of astronomers like Galileo, Newton was studying the motion of planets. As part of this study, Newton was trying to pin down the value of the slope of a curve. This was easy: it is the ratio of rise to the run. However, in the case of a celestial object like a planet, this slope is constantly changing. There was no easy method to finding the exact slope at a given point in time. So, Newton developed a new theory, that he termed the study of fluxions, and what we know today as calculus.

Newton was not sure if his theory – the derivative of a function – would find acceptance from his peers. He kept his ideas to himself and shared them only with a few of his friends. If even Newton could suffer from such insecurities, it is easy to understand our own trepidations. Even though it was not Newton's goal to shake the world, calculus has made an impact on so much of our life that no one graduates from high school without, at least, an introductory course in calculus.

Euler [2]

The year is 1736 and Königsberg is a prosperous town in Russia. The river Pregel runs through the city creating two sides, say A and B. Additionally, the river has created two islands between A and B, say C and D, as it curves through Königsberg. There are seven bridges that connect C and D to each other and to A and B on either side of the river (see Figure 1). A story goes that the mayor of Königsberg enjoyed taking a walk through the city every Sunday. To make his walk more interesting the mayor tried to design it so he would cross each of the seven bridges, but only once. No matter how hard he tried, he found that to be impossible – there was always a bridge that he needed to cross a second time to return home. In his frustration, he wrote to the famous mathematician Leonard Euler who lived in the nearby city of St. Petersburg and sought his help.

Initially, Euler dismissed the problem as trivial because it bore no connection to mathematics but was nonetheless still intrigued by it. He modeled the problem as a graph – a concept that was not invented yet. The landforms, A, B, C,

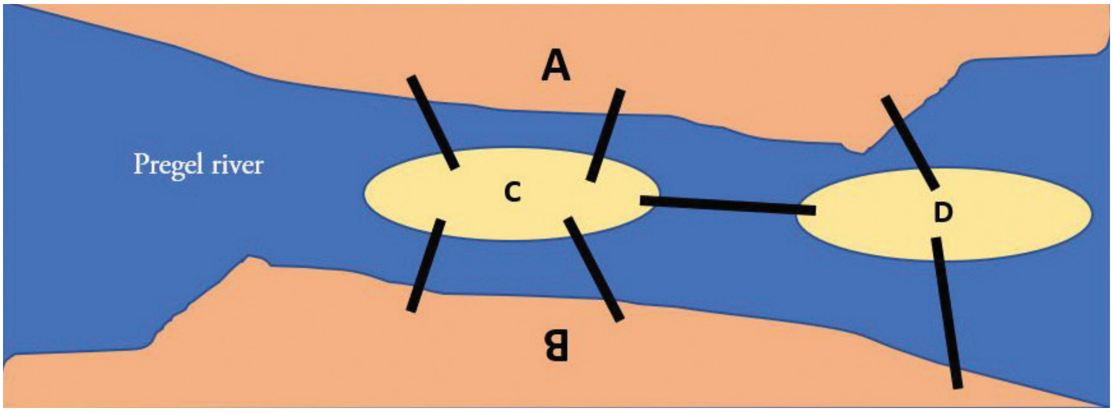

Figure 1

D represented nodes of the graph and the seven bridges connecting them became the edges (see Figure 2). Euler proved that in order to cross each edge just once, the graph must contain either zero or two nodes that have an odd number of edges that lead into them. Because the Königsberg graph had four nodes with an odd number of edges leading into them, it was impossible to come up with a walking path that crossed each bridge just once.

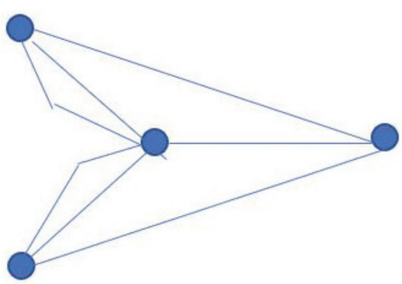

Figure 2

Euler's discovery led to the development of graph theory and topology, and today graph theory has permeated so many branches of knowledge like computer science (graph databases), sociology (social network analysis), linguistics (language analysis), biology (protein and DNA analysis), transportation (shortest path problem), and national security (terrorist and financial networks).

Von Neumann [3]

Poker is a card game that relies on your ability to bluff and deceive your opponents into believing that you hold a stronger hand than them. In 1929, a brilliant young mathematician, Jon Von Neumann, moved to the United States from Germany to take a position at Princeton University. Von Neumann liked playing poker but was not particularly good at it. Poker inspired Von Neumann to formalize the theory of bluffing into what is known today as Game theory.

Game theory seeks to provide a foundation for instances involving multiple parties that make decisions based on predicting the moves of the opposing parties. Situations involving competing players are modeled to produce strategy or help reach an optimal decision. Game theory is a widely used tool in economics. A testament to the power of Game theory is the many Nobel prizes awarded to economists who have applied game theory principles and its refinements to aspects of economics. 'A Beautiful Mind' is a must-see film made on the life of one such Nobel prize winning Game theory economist, John Nash. Game theory finds use in many other fields like computer science (networks and resource allocation), biology (coexistence of wildlife), government (auditing taxpayers, warfare), psychology and, of course, politics.

In all fairness, Von Neumann had an advantage over most other people: like Ramanujan, he was a child prodigy. He possessed a photographic memory, spoke fluent Greek, and by the age of eight was familiar with calculus! So, while Von Neumann may not be the best motivational figure, this story serves to impress upon the reader that mathematicians hold the power to make a big impact on the world.

You may question the fact that the most recent example given is still from a century ago. Aren't there any more recent examples of mathematical discoveries that have 'shaken' the world? The reason behind this is the lag between making a discovery and finding a problem to which it can be applied. It takes time before a result can find its way into other areas. Do not let that discourage you. That is partly the reason why most Nobel prizes awarded are for discoveries the recipients made several years before the award recognized them.

To Solve Common Problems

If shaking the world is not your thing, there still are accomplishments that you as a mathematician can make that will advance a field and help humankind make progress. In this section I shall present three examples of mathematics applied to real world problems that are helping advance our society.

Cryptography [4] (1977)

Cryptography, or the art of obscuring a message so only the intended parties can understand it, is not new. Even Julius Caesar used it for communicating with his generals. However, finding ways to encrypt and decrypt messages that even computers cannot break is challenging. But mathematics, again, came to the rescue of the world.

Whitfield Diffie, Ron Rivest (R), Adi Shamir (S), and Leonard Adelman (A) – all with basic degrees in mathematics – along with Martin Hellman and Ralph Merkel designed the public key cryptographic system to encrypt and decrypt messages. The RSA algorithm uses the concept of prime factors to generate a one-way function (a function that is easy to implement in one direction but almost impossible to perform in the reverse direction) to make this happen. The basic premise is this: you are given a very large number and the challenge is to find two prime factors of that number. If the answer (the two primes) is given to you, then finding their product is trivial, but going the other way is almost impossible for humans and even the fastest computers will take years to find the two primes.

It was not until the invention of the Internet, almost twenty years later, that public key cryptography found widespread use (the lag I mentioned previously). All the applications we use today and take for granted when we browse the Internet, make phone calls, send messages, protect our files, etc., are the result of the efforts of mathematicians applying their skills and knowledge to common problems.

An interesting fact is that a British mathematician, Clifford Cocks, while working for the UK spy agency GCHQ had already developed the idea of public key cryptography. The British government kept the invention classified until finally acknowledging it 24 years later in 1997. Rivest, Shamir, and Adelman independently developed the same result and received the fame and patents that came with it.

Cricket [5] (2000)

When I was a young boy growing up in India, cricket was the only thing that interested me. I could not get enough cricket. If listening to cricket commentary on the radio could nourish a body, I would easily skip lunch. Cricket had one point of universal frustration: the Leg-Before-Wicket (LBW) decision – there was no way to tell if the umpire's decision was correct: that, if not impeded, the ball would have hit, or not hit, the wickets. Good sportsmen usually begrudgingly accepted the umpire's decision if it went against them.

Paul Hawkins, a British student of mathematics and artificial intelligence, faced a similar bad umpiring call during a cricket match. But instead of just getting mad, he decided to do something about it. He invented Hawk-Eye, a system that uses high speed cameras and a complex computer program to predict the path of a ball. It was the cricket commentators who initially used Hawk-Eye to determine if the umpire got it right. TV viewers saw the projected path of the ball superimposed on the image of the game. Cricket governing bodies soon adopted Hawk-Eye as an official tool for use during appeals and times when the umpire is unable to make the call. Sony bought the Hawk-Eye system from Hawkins for millions of dollars. Today, Hawk-Eye has found use in other sports like tennis and soccer.

So how does Hawk-Eye use mathematics? Hawk-Eye uses concepts from Epipolar geometry, linear algebra, and statistics. Many high-speed cameras, separate from the TV cameras, record the ball and then compute its most likely path. Epipolar geometry helps by recognizing corresponding points on two separate images of the same object. Triangulation is the process of identifying the location of a point in 3D space given its location in two or more 2D spaces. In this case, each frame from every camera contains the picture of the ball in 2D space. The system takes that position from one camera and compares it with the position of the ball from other cameras for the same frame. Triangulation then computes the 3D coordinate of the ball. Triangulation is repeated for each frame yielding a set of 3D points that form the path of the ball. Finally, the projected path of the ball is computed to determine if it will hit an object in that path, like the wickets.

Natural Language Processing [6] (2013)

Natural Language Processing (NLP) is the ability of computers to understand human language, be it written or spoken. The last few years have seen incredible advances in this field. Google, Amazon, Apple have products like 'OK Google', Alexa, and Siri that allow us to speak commands to perform actions. You may have noticed Gmail making suggestions for completing your sentences as you type your email. In this case, based on the context of your sentence, Gmail is predicting your next set of words.

Neural Networks are at the center of much of the advances in NLP. While neural networks themselves are based on mathematical concepts of linear algebra and calculus, for this topic, I am going to focus on one aspect of NLP: word embeddings.

Consider the following two sentences:

- I had a wonderful time at the party
- I had a great time at the party

To us, the two sentences express very similar sentiments. However, for a computer the words 'wonderful' and 'great' are very different and making the inference that the two sentences are very close semantically is very difficult. This is because neural networks take as inputs numbers and not characters or words. So how do we represent the words as numbers and then how can we compare their closeness? The answer is word embeddings. Simply put, word embeddings are vector representation of words as in the example below:

Wonderful: [12, 17, 15, 19, 27, 53]

Great: [12, 16, 16, 22, 25, 60]

The words Wonderful and Great are represented as vectors of 6 numbers each (or a 6 dimensional vector). Now that words are represented by vectors, how do we tell if these words are "close" in meaning? Mathematics, once again, provides the solution: cosine similarity. Cosine similarity or cosine distance is a measure of how close two words are and is mathematically computed as the cosine of the angle between the two multi-dimensional vectors. The smaller the distance, the closer the words.

You are probably wondering how I came up with the values in the vectors. In my example, the numbers are made up to illustrate the point. However, from an implementation standpoint, a group of Google researchers led by Czech computer scientist Tomas Mikolov invented the word2vec algorithm in 2013 that generates the vector for a given word.

To Become Rich and Famous

There are many opportunities to achieve fame and fortune through Mathematics. Here are four such avenues.

The Clay Mathematics Institute[7] has identified seven problems, called The Millennium Prize Problems, which have intrigued mathematicians in recent times. The Clay Institute considers these problems sufficiently hard to warrant a prize of $1M (US) per problem for a solution. To date the only Millennium Prize Problem solved is the Poincare Conjecture (from the field of topology, thank you Euler!). The winner of this award, a Russian mathematician named Grigori Perelman, declined to accept the prize money.

If you were wondering why there is no Nobel prize in mathematics, you are not alone. In 1899 when the Swedish philanthropist Alfred Nobel did not create a prize in mathematics, a Norwegian mathematician Sophus Lie proposed to create one. The monarch, who at that time was the king of both Norway and Sweden, agreed to fund the award for mathematics. Unfortunately, Norway and Sweden decided to dissolve their union under a common regent before the prize was instituted. It was not until 2002 that the Government of Norway finally established the Abel prize for contributions to mathematics.

The Abel [8] prize is the Nobel equivalent for mathematics and the cash award is comparable. While the Royal Swedish Academy picks the Nobel winner, the Norwegian Academy of Sciences and Letters picks the winner of the Abel. Remember the Nobel Prize winning Game theory economist John Nash? He won the Abel in 2015. One Indian, S.R. Srinivas Varadhan Iyengar, won the Abel in 2007.

The Fields Medal [9] is another prestigious honor in mathematics. The International Mathematical Union chooses the winners once every four years. Candidates for this award must be under 40 years and are recognized not just for their outstanding contributions to mathematics but also for showing promise of future accomplishment.

The Putnam Competition [10] is an annual mathematics test that is open to undergraduate college students in the USA and Canada. Organized by the Mathematical Association of America this is considered by many to be the toughest mathematics test in America and in the world. There are 12 questions worth 10 points each for a total of 120 points. The test is so hard that the median score is between 0 and 1! The prize money is not high, but the prestige is enormous: the top 5 students are designated Putnam Scholars. These students get scholarships to some of the top universities. Many employers look to steal these scholars by making very generous offers.

With a love for mathematics coupled by hard work and dedication, it is not only possible to become famous and find lucrative jobs, but also win enough money to retire from that lucrative job!

To Become a Problem Solver

A mathematical model is a representation of a system using mathematical symbols, objects, and operators. Solving a real-world problem starts with modeling the problem: stating the problem in a mathematical format. Once you have a model, you apply mathematical principles to solve that problem.

A mathematics education will teach you clarity of thought and good problem-solving skills. More formally, a mathematics education will help you gain expertise in developing mathematical models from problem descriptions, then apply mathematical principles to solve those problems (or as in the case of the Königsberg problem, show that the problem has no solution). It follows that this incredible skill will prove useful in almost any situation, even a domestic one as I will now illustrate. This is a problem that I faced just a few weeks ago.

I want to fertilize the lawn in my front yard. The fertilizer comes in liquid form and I have a spray bottle by which I can apply the fertilizer to my lawn. Here are the basic steps:

1. The spray bottle dispenses 2 litres of water per minute
2. A dial on the spray bottle can be set to mix from 1 tsp to 10 tsp of fertilizer per litre
3. Fill the spray bottle with the required quantity of fertilizer and adjust the dial
4. Attach the water hose to one end of the spray bottle
5. Open the tap and when the water starts flowing, spray the lawn from the other end of the spray bottle
6. Fertilizer from the spray bottle will mix with the water from the hose and feed the lawn.

Question: How much liquid fertilizer do I add to the spray bottle and what dial setting should I use? I do not want to overfeed my lawn and at the same time I do not want to pour back any remaining fertilizer in the spray bottle back into the fertilizer container. Ideally, when I have fertilized the entire lawn, I want the spray bottle to be empty.

Here are the other facts you will need:

1. The area of the lawn in 2000 ft^2
2. The fertilizer instructions say to use 3 tsp mixed with 3 litres of water per 1000 ft^2
3. It takes me about 1.5 minutes to walk from one end of the lawn to the other

My solution was to fill the sprayer with 6 tsp of fertilizer, set the dial to 1 tsp, and cross my lawn twice. You can verify my solution. If it is incorrect, then you know I did not graduate with a mathematics degree. My lawn, however, is doing well.

To Love the Subject for the Sake of the Subject

It is not necessary to love a subject because it offers job opportunities and the promise of big money. Mathematics can keep an interested student happy with a lifetime of problems that have absolutely no bearing on real life. Mathematicians enjoy the subject for its sheer beauty.

One such beautiful flower from the garden of mathematics is the one proposed by French mathematician Pierre de Fermat (Fermat's Last Theorem). He wrote in the margins of a book he was reading: there are no three positive integers X, Y, Z that can solve the following equation:

$$X^n + Y^n = Z^n \text{ for values of } n > 2.$$

An uncorroborated story is that Fermat claimed he knew of a proof but there wasn't enough room in the margin to write it out. Fermat wrote this in 1637 and for over 350 years no one was able to "find" that proof that Fermat thought he had. This conjecture finally became a theorem in 1995 when British mathematician Andrew Wiles, working at Princeton University, supplied the proof. However, the techniques that Wiles used in his proof were not discovered at the time of Fermat.

Another problem that kept mathematicians engaged for a long time was the one stated by South African mathematician Francis Guthrie in 1852. He was coloring a map of the counties in England and noticed that only four colors were sufficient to complete the task. His observation turned into the four-color conjecture: only four colors are sufficient to color a map to ensure that no two adjacent regions are colored with the same color. The proof eluded mathematicians for over 145 years and when the proof was finally published, it did not use maps at all; in fact, it used graphs! Once a proof is proposed, it needs to be verified. This was the first proof that used a computer assisted technique. This technique helps verify the correctness of the proof in a much shorter period than hand, paper, and pencil techniques. Not all mathematicians accept this method of proving theorems.

Two Indian mathematicians deserve special mention under this section for loving mathematics for the sake of the subject. One is the prodigy Srinivasa Ramanujan [11] who lived in the late 19th century - Ramanujan whose contributions are so profound that scholars are still trying to make sense of his work. So deep was his work that it is said that had Ramanujan not lived, nobody would have discovered many of the things that he did. Ramanujan worked very differently from the traditional mathematicians of that time and of today. Traditionalists begin by stating proven results, then extending them, carefully justifying each step. Ramanujan's approach was different: for him Mathematics was pure divinity where concepts just came to him like water from a hose; practicality did not matter, nor did formal proofs. In my opinion, the need for "providing methodical proofs" may even have dampened his capacity for producing more of what he eventually did.

The other is D.R. Kaprekar [12] who was a math schoolteacher in rural India. My father took me to several of Kaprekar's lectures. His enthusiasm for the subject was palpable. He was a man of very modest means: his apparel consisted of a worn white Indian-style *dhoti* held together by a rope, a shirt, and a jacket. He loved numbers and, in his lectures, showed us many interesting properties of numbers that he had discovered. He put in a lot of labor in playing with numbers to find enthralling qualities about them. The number 6174, named after him, is known as Kaprekar's constant. It has a property derived from the following rules:

1. Take any four-digit number, not all digits being equal (e.g., 1234)
2. Rearrange the digits to create two four-digit numbers: one the largest possible and one the smallest possible with those digits (4321 and 1234)
3. Subtract the smaller from the larger (3087)
4. Go back to step 2 and repeat.

This process will, in at most 7 iterations, always reach the constant 6174. Clearly, neither the number nor the process has any real-world implications. Also, where do you begin to find an interesting characteristic like this? Questions such as these did not dissuade Kaprekar; he found beauty in numbers and numbers engaged him for many years. Confessing his addiction for numbers, he would say: "A drunkard wants to go on drinking wine to remain in that pleasurable state. The same is the case with me in so far as numbers are concerned."

Mathematics has the quality to engross you for years and it is yours for the taking. Numbers, anyone?

Conclusion

Galileo has said that "Mathematics is the language in which God has written the universe." Logic tells us that mastering this language will help you understand and decode the universe. Mathematics has the power to discover knowledge that will pave the way for our society's growth and progress. The myth about job opportunities can now be busted: many companies like Google, Facebook, Amazon, Tesla, SpaceX, and organizations like NASA are hungry for graduates with degrees in mathematics. All the technological advances that are happening around us are a result of applying mathematical techniques, results, and models to common problems. Mathematics is the foundation on which these advances rest and it makes sense to hire mathematicians to understand and build this foundation.

My one regret in authoring this essay is not including achievements of women in mathematics. It is unfortunate that only men have been the major players in the history of mathematics. Another amazing film 'Hidden Figures' tells the story of African-American women mathematicians (called computers because they performed computations) who worked at NASA during the years of the space race. While the movie celebrates those women computers, it also points out discrimination they faced, both from men who thought women were not good enough, and from whites who thought black people were not good enough. Our society is evolving, and more women are taking on challenges traditionally tackled by men. Going forward, women will write many of the stories of developments in mathematics. I just cannot wait for that.

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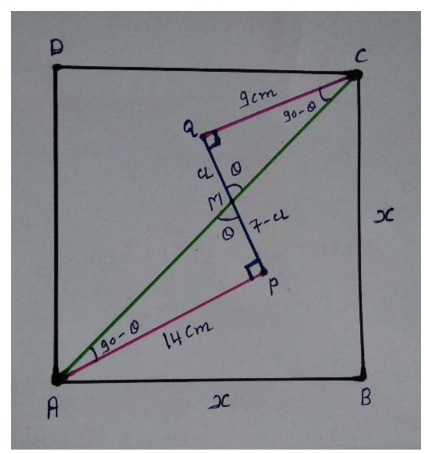
Think out of the box!

Problem on page 68, July 2020 issue:

Find side x of the Square

Solution: As shown in the figure, Let **QM = a** and **PM = 7 – a**. Now we know that ΔAPM and ΔCQM are similar triangles.



$$\therefore \frac{\mathbf{CQ}}{\mathbf{AP}} = \frac{\mathbf{QM}}{\mathbf{PM}} = \frac{\mathbf{CM}}{\mathbf{AM}}$$

$$\therefore \frac{9}{14} = \frac{a}{7-a} = \frac{CM}{AM} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore a = \frac{63}{23}$$

Now $\mathbf{PM} = 7 - a = 7 - \frac{63}{23} = \frac{98}{23}$

$$\therefore \mathbf{PM} = \frac{98}{23}$$

According to the Pythagoras theorem, we can write

$$AM^2 = AP^2 + PM^2 = 14^2 + \left(\frac{98}{23}\right)^2 = \frac{113288}{529}$$

$$AM = \frac{238\sqrt{2}}{23}$$

Now $CM = \frac{9}{14}\,AM = \frac{9}{14}\left(\frac{238\sqrt{2}}{23}\right) = \frac{153\sqrt{2}}{23}$

Again, according to the Pythagoras theorem

$$x^2 = \frac{AC^2}{2} = \frac{(AM+CM)^2}{2} = \frac{\left(\frac{238\sqrt{2}}{23} + \frac{153\sqrt{2}}{23}\right)^2}{2} = 289$$

$$\therefore x = 17$$

Solution sent in by Tejash Patel, Chanasma Primary School No.2, Patan, Gujarat.

For another Solution please see page 86