Extension of an INMO Geometry Problem

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In this work, I extend the result contained in an INMO (Indian National Mathematics Olympiad) geometry problem. This problem refers specifically to a right-angled triangle, but I prove a general result that holds for any triangle. The INMO result is then a special case of this.

Problem (INMO 2016–5). *Let ABC be a right-angled triangle with $\measuredangle B = 90^\circ$. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of the triangle ABC, prove that*

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

The result that we shall prove is the following.

Theorem. *In triangle ABC, let D be a point on side AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of the triangle ABC, then*

$$\frac{1}{r'} = \frac{1}{r} + \frac{2}{\sqrt{(a+b-c)(b+c-a)}}.$$

Proof of the general result

Let ABC be any triangle, and let D be a point on AC such that the inradii of the triangles ABD and CBD are equal, say r'. Let the inradius of triangle ABC be r. Let the areas of $\triangle ABC$, $\triangle ABD$ and $\triangle CBD$ be Δ, Δ_1 and Δ_2 respectively. (See Figure 1.) Let the semi-perimeters of $\triangle ABC$, $\triangle ABD$ and $\triangle CBD$ be s, s_1 and s_2 respectively. Let $BD = d$, $AD = x$, $CD = b - x$. Also let $\measuredangle ABD = \theta$.

Keywords: Inradius, sine rule, cosine rule

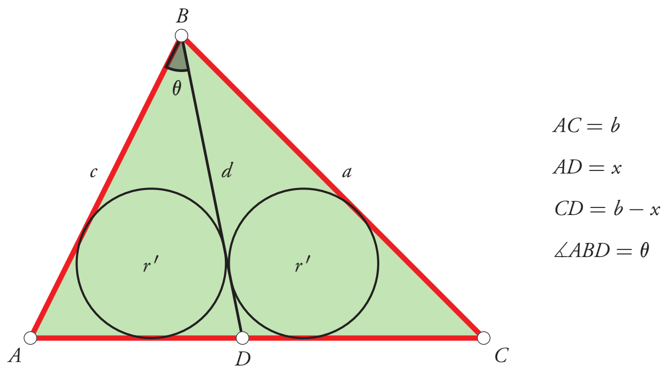


Figure 1. Triangles *ABD* and *CBD* have equal inradii (r')

Then we have,

$$\Delta_1 = r' s_1, \tag{1}$$

$$\Delta_2 = r' s_2. \tag{2}$$

Now, from (1) and (2), we get

$$\frac{\Delta_1}{\Delta_2} = \frac{s_1}{s_2}. \tag{3}$$

Using the fact that for a pair of triangles with the same height, the ratio of their areas is equal to the ratio of the bases, we get from (3),

$$\frac{x}{b-x} = \frac{c+d+x}{a+d+b-x}. \tag{4}$$

From (4) we get:

$$x = \frac{b(c+d)}{a+c+2d}, \tag{5}$$

$$b - x = \frac{b(a+d)}{a+c+2d}. \tag{6}$$

Again, from (1) and (2) we get

$$\Delta_1 + \Delta_2 = r'\left(s_1 + s_2\right). \tag{7}$$

Since $\Delta_1 + \Delta_2 = \Delta$ and $\Delta = rs$, this becomes $rs = r'\left(s_1 + s_2\right)$, i.e.,

$$r(a+b+c) = r'(a+b+c+2d). \tag{8}$$

From this we get, by division:

$$\frac{r}{r'} = 1 + \frac{2d}{a+b+c} = 1 + \frac{d}{s},$$

$$\therefore \frac{1}{r'} = \frac{1}{r} + \frac{d}{rs},$$

$$\therefore \frac{1}{r'} = \frac{1}{r} + \frac{d}{\Delta}. \tag{9}$$

Now, applying the sine rule in $\triangle ABD$, we get

$$\frac{d}{\sin A} = \frac{x}{\sin\theta}. \tag{10}$$

Applying the sine rule in $\triangle ABC$ and using (5),

$$\sin\theta = \frac{a(c+d)\sin B}{d(a+c+2d)} \tag{11}$$

Similarly, applying the sine rule in $\triangle CBD$ and using (6), we get

$$\sin(B-\theta) = \frac{c(a+d)\sin B}{d(a+c+2d)}. \tag{12}$$

Expanding $\sin(B-\theta)$ and using (11), we get

$$\cos\theta = \frac{c(a+d)+a(c+d)\cos B}{d(a+c+2d)}. \tag{13}$$

Now, squaring and adding (11) and (13), we get

$$d^2(a+c+2d)^2 = a^2(c+d)^2\sin^2 B + [c(a+d)+a(c+d)\cos B]^2$$

After simplifying, it becomes

$$2d^2 = ac(1+\cos B) \tag{14}$$

Now, applying the cosine rule for $\measuredangle B$ in (14), we get the following very nice result:

$$4d^2 = (a+c)^2 - b^2. \tag{15}$$

Now, putting

$$\Delta = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{b+c-a}{2}\right)},$$

$$d = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)},$$

in (9), we get the desired result

$$\frac{1}{r'} = \frac{1}{r} + \frac{2}{\sqrt{(a+b-c)(b+c-a)}}. \tag{16}$$

Solution to INMO problem using the theorem

The INMO problem is a special case of our general result. Here $\triangle ABC$ is a right-angled triangle with $\measuredangle B = 90^\circ$.

Taking $\cos 90^\circ = 0$ in (14) or $a^2+c^2=b^2$ in (15), we get $d^2 = \frac{1}{2}ac$, but for a right-angled triangle ABC, $\Delta = \frac{1}{2}ac$. Hence,

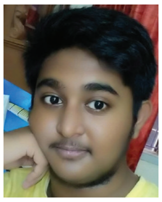
$$\Delta = d^2.$$

Putting this in (16), we get, as desired:

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{d}.$$

References

1. M.R. Modak, S.A. Katre, V.V. Acharya, V.M. Sholapurkar, *An Excursion in Mathematics*, Bhaskaracharya Pratisthana, Pune (2017)



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