The N-Queens Problem

DEENS ACADEMY
MATH CLUB

Introduction

The n-queens problem is a generalization of the eight-queens problem of placing eight queens on a standard chessboard so that no queen attacks any other queen. The original eight-queens problem was first posed in 1848 by Bezzel, a German chess player, in the *Berliner Schachzeitung* (or the Berlin *Chess Newspaper*). The generalization is due to Linolet, who asked the same question later in 1869, but now for n queens on an n x n board.

Before delving into this interesting problem, let us recall that a queen placed on a square on a chessboard can attack any other piece that is in its line of sight either horizontally, or vertically, or diagonally. See Figure 1. Thus, to solve the n queens problem, we need to place these n queens on an n x n chessboard such that there is exactly one queen in each row and column, and to ensure simultaneously that no two queens lie on the same diagonal.

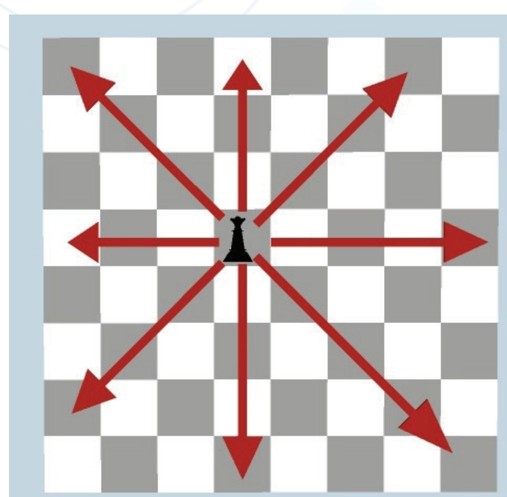

Figure 1. A Queen placed as shown can attack any other piece that is in its line of sight along the arrows shown: horizontally along its own row, vertically along its own column, or along the two diagonals leading out from it.

Keywords: games, strategy, reasoning, logic, mathematics

Nauck in 1850 was the first to give all 92 solutions to the original 8 queens problem. (That there are indeed exactly 92 solutions to the 8 queens problem was claimed by Gauss in a private communication, and proved by Pauls in 1874.)

Pauls was the first to solve the n queens problem for all $n > 3$. (For $n = 2,3$ there are no solutions, as is readily observed.) His solutions were followed by several other solutions for all n, including 1) Franel, 2) Scheid, 3) Hoffman, Loessi and Moore, 4) Falkowski and Schmitz, 5) Yaglom and Yaglom, to name a few. Several others gave solutions when n was restricted to lie in various subsets of the positive integers.

The total number of solutions for n from 4 to 11 are 2,10, 4, 40, 92, 352, 724 and 2680 respectively, and this was already known to Sprague in 1899. For larger n, the *backtracking* algorithm can be used to generate all solutions.

The problem still retains much fascination, and continues to be studied. Why study this problem if it has already been solved? It was initially studied for "mathematical recreation." However today, the problem is applied in parallel memory storage schemes, VLSI testing, traffic control and deadlock prevention in concurrent programming. Other applications include neural networks, constraint satisfaction problems, image processing, motion estimation in video coding, and error correcting codes. Additionally, the problem appears naturally in biology, where it was observed that the computation involved in the analysis of the secondary structure of nucleic acids is analogous to that involved in finding solutions to the n-queens problem!

We recommend the paper [1] for a comprehensive survey of the n-queens problem.

Our Solutions

In this article, we will present one solution to the n-queens problem, for all $n > 3$, that we the members of the Deens Academy Math Club obtained independently. We were unaware of previous solutions while we worked on this problem (and indeed were unaware then of the paper [1]), but were pleased to discover later that known solutions contain features of our solution.

We consider squares of size n x n, and write n as $2p$ when n is even, and as $2p + 1$when n is odd. We will divide our solution into two cases: the first where p is congruent to 0 or 2 mod 3 (for example, $p = 6, 9,12$, etc. or $p = 5, 8, 11$, etc.), and the second where p is congruent to 1 mod 3 (for example, $p = 4, 7, 10$, etc.).

We will index our squares by the row and the column on which they are situated: for example, a queen with indices (a, b) indicates that it is on the a-th row and b-th column. We will count the topmost row as row 1 and the bottommost as row n; similarly, we will count the leftmost column as column 1, and the rightmost as column n.

Of course, two squares (a, b) and (c, d) are on the same row if and only if $a = c$ and on the same column if and only if $b = d$. Elementary coordinate geometry shows that two squares $(a, b)$and (c, d), not both on the same row (so $c \neq a$), are on the same diagonal if and only if $\frac{d-b}{c-a} = \pm 1$. See Figure 2. We refer to the expression $\frac{d-b}{c-a}$ as the *slope* between the two squares. Recalling the orientation of our coordinate system: $(1, 1)$ is at the upper left and (n, n) is at the bottom right – we see that if the slope between (a, b) and (c, d) is 1 then they are on a diagonal that runs in the Southeast/Northwest direction, and if the slope is -1 then they are on a diagonal that runs in the Southwest/Northeast direction.

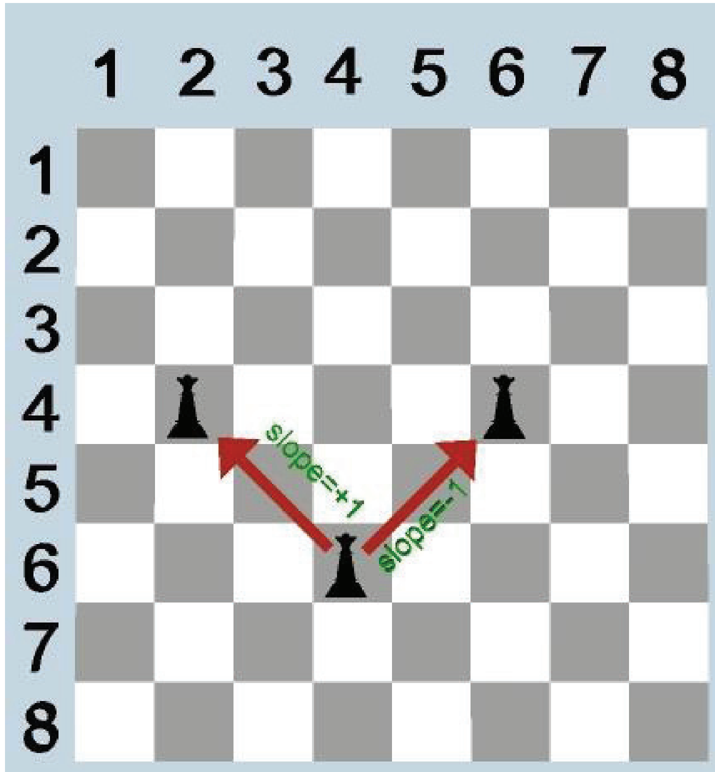

Figure 2. Slope between squares on diagonals: The slope between the squares $(4, 2)$ and $(6, 4)$ is $\frac{4-2}{6-4} = 1$. The slope between the squares $(6, 4)$ and $(4, 6)$ is $\frac{6-4}{4-6} = -1$.

Case A: p congruent to 0 or 2 mod 3

We will first provide the solution for the case $n = 2p$; the solution for the case $n = 2p + 1$ will then be just a simple modification. (The smallest value of p here is 2, corresponding to a 4×4 board.)

We place the queens on squares with the indices $(i, 2i)$ for the upper half of the square, so $1 \le i \le p$. (See Figure 3.) We then rotate this arrangement by 180 degrees about the geometric center of the chessboard to place the queens on the lower half of the square. It is easy to see that the reflected position of the square $(i, 2i)$ is $(n + 1 - i,\ n + 1 - 2i)$, so the queens in the bottom half have coordinates $(n + 1 - j,\ n + 1 - 2j)$, with j once again in the range $1 \le j \le p$. (For example, the rotation of the square at $(8, 16)$ in Figure 3 is the square at $(9, 1) = (17 - 8, 17 - 16)$.) We notice that as j decreases from p to 1, the row index of the rotation, namely $n + 1 - j = 2p + 1 - j$, increases from $p + 1$ to $2p$. Similarly, as j decreases from p to1, the column index of the rotation, namely $n + 1 - 2j = 2p + 1 - 2j$, increases from 1 to $2p - 1$ in steps of 2.

We need to show that no two queens are on the same row, or on the same column, or on the same diagonal. It is clear that two queens cannot be on the same row, as the row indices run from 1 to p in the placements in the upper half and from $p + 1$ to $n = 2p$ in the bottom half. The queens in the upper half are on distinct *even numbered* columns $2i$, $1 \le i \le p$, while those on the bottom half are on distinct odd numbered columns $1, 3, \ldots, 2p - 1$. Thus, all queens are on distinct columns.

To show that no two queens are on the same diagonal, positive or negative, we consider the following cases separately:

Subcase 1. Both queens are on the top half. Quite visually, the queens are not on the same diagonal (see Figure 3 for example), but we can see this formally as follows: since they are both on the top half, one

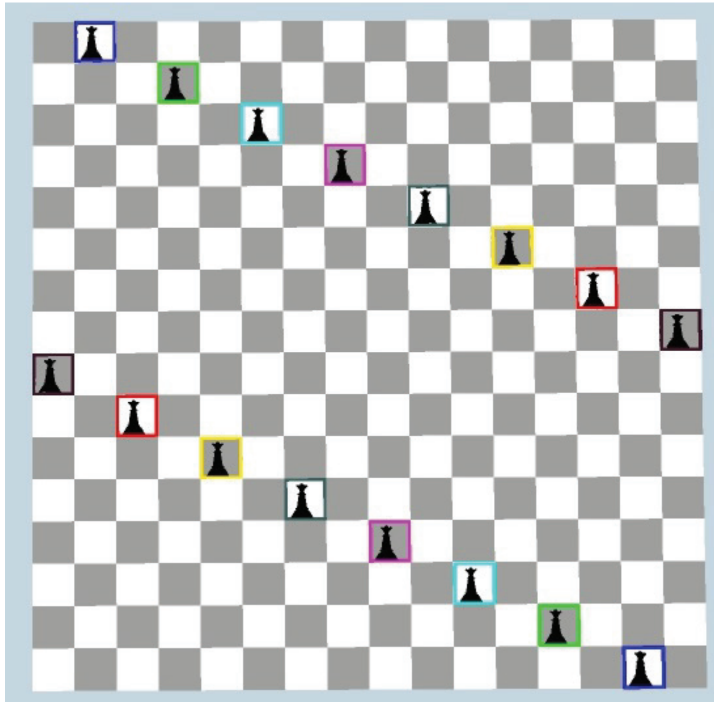

Figure 3. Arrangement when p = 8. The squares that correspond to each other after rotation are shown in the same colour.

queen will have coordinates $(i, 2i)$and the other will have coordinates $(j, 2j)$, where i and j are distinct. If say i <j, then i and j satisfy $1 \leq i < j \leq p$. The expression for the slope shows that the slope between the pair of points is $(2j - 2i)/(j - i) = 2 \neq \pm 1$, so they are not on the same diagonal.

Subcase 2. Both queens are on the bottom half. This can be proved just as in Case 1, or by observing that Case 2 follows from Case 1 by symmetry: if two queens on the bottom half were on the same diagonal, then their rotations in the top half will be on the diagonal that is the rotation of the first one, and we have already shown above that this cannot happen.

Subcase 3. One queen from the top half and the other from the bottom half. The queen on the top half has coordinates $(i, 2i)$ for some i in the range $1 \leq i \leq p$, while the queen on the bottom half has coordinates $(n + 1 - j,\ n + 1 - 2j) = (2p + 1 - j, 2p + 1 - 2j)$ for some j in the range $1 \leq j \leq p$. The slope is thus given by $(2p + 1 - 2j - 2i)/(2p + 1 - j - i)$. If this slope were 1, we would find on cross multiplying that $i + j = 0$, which is impossible as i and j are both positive. If the slope were -1, we would find on cross multiplying that $4p + 2 = 3(i + j)$. Thus $4p + 2\ (= 3p + p + 2)$ is congruent to 0 mod 3, so $p + 2$is congruent to 0 mod 3, or what is the same thing, p is congruent to 1 mod 3. But this violates our assumption that p is congruent to 0 or 2 mod 3, and therefore, the slope cannot be -1 either.

We have thus proved that our arrangement of queens for the case $n = 2p$, p congruent to 0 or 2 mod 3, indeed satisfies the requirement of the puzzle. Finally, when $n = 2p + 1$, with p congruent to 0 or 2 mod 3, we place the queens in rows 1 through $2p$ exactly as described above, and place the $(2p + 1)$-th queen in the square with coordinates $(n,\ n) = (2p + 1, 2p + 1)$. See Figure 4. It is clear that the row and column indices of the squares in the first $2p$ rows are different from $2p + 1$, so this last queen does not share a row or column with the previous $2p$ queens. The only diagonal that this last queen lies on is the "main diagonal" of the square which runs from $(1, 1)$ to (n, n). This diagonal has the property that for any square on it, the row and column indices are equal. It is clear that none of the previous queens are on this diagonal as for each of them, the row and column indices — i and $2i$ in the upper half and

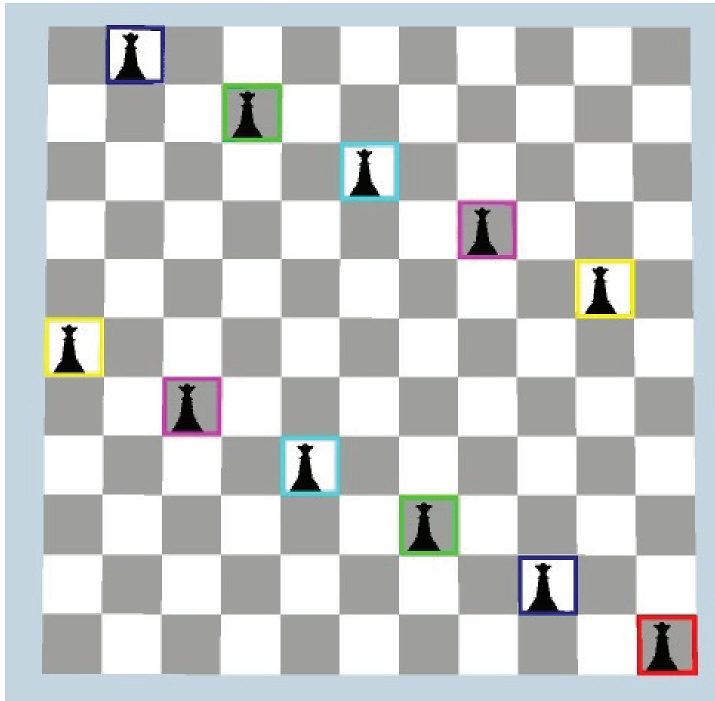

Figure 4. The case of the 11×11 square: here $11 = 2p + 1, p = 5$, and 5 is congruent to 2 mod 3. The solution is built from the 10 X 10 case $(10 = 2p, p = 5)$ by adding a queen at the bottom right square.

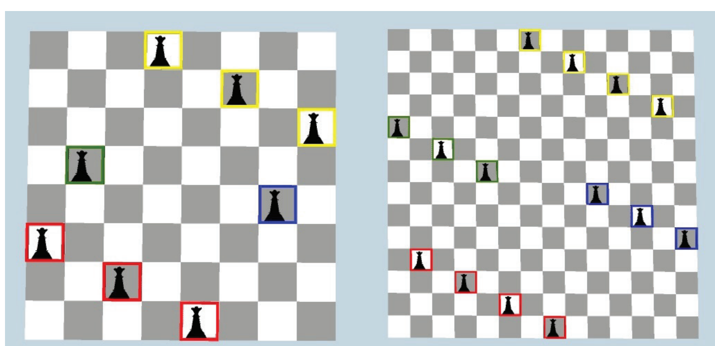

Figure 5. Arrangements when p = 4 (even) and p = 7(odd)

$2p + 1 - j$, $2p + 1 - 2j$ in the bottom half — are unequal. Thus, this arrangement for the case $n = 2p + 1, p$ congruent to 0 or 2 mod 3, indeed satisfies the requirement of the puzzle.

Case B: *p* congruent to 1 mod 3

As in Case A, we will first provide the solution for the case $n = 2p$; the solution for the case $n = 2p + 1$ will then be the same modification as before.

We arrange the queens as shown in Figure 5.

We will suggestively label the four sets of queens shown as Y, G, R, B. In the set Y, we start at the square $(1, p)$, and move in increments of two squares to the right and one square down, till we cannot move any more without falling off the right side of the board. We continue to the set G by "wrapping the pattern in Y around to the left side of the board. Thus, if the last queen in Y is in position (a, b), then the first queen in G has position $(a + 1, (b + 2) \ mod \ n)$, the second has position $(a + 2, (b + 4) \ mod \ n)$, and so on. We stop adding queens to G when the next queen to be added attacks a queen in Y. The sets R and B are then obtained by simply rotating the half of the chessboard containing the sets Y and G by 180 degree about the geometric center of the chessboard as we did in Case A. It is easy to see that the indices for the queens in the various sets are as follows:

Y	$(i,\ p+2(i-1))$	i from 1 to $(p+1)/2$, when p is odd 1 to $(p+2)/2$, when p is even
G	$(k, 2k-p-2)$	k from $(p+3)/2$ to p, when p is odd $(p+4)/2$ to p, when p is even
R	$(2p+1-j, p-2j+3)$	j from 1 to $(p+1)/2$, when p is odd 1 to $(p+2)/2$, when p is even
B	$(2p+1-l, 3p-2l+3)$	l from $(p+3)/2$ to p, when p is odd $(p+4)/2$ to p, when p is even

Table 1: (Row, Column) indices of the queens in Case B.

As in Case A, we compare different queens and show they are not on the same row, or column, or diagonal. It is clear from the row indices that the queens are all on different rows. As for columns, it is clear from the column indices that *within* each of the sets Y, G, R and B, the queens are in different columns. Similarly, the column indices show that every queen in Y is on a different column from every queen in G. By symmetry, the same is true for every queen in R and every queen in B. To compare, for instance, the column of a queen in Y and a queen in R, we notice that when p is even, the column indices of queens in both Y and G are even, while the column indices of queens in R and B are odd. When p is odd the situation is reversed, the queens in Y and G are on odd columns while those on R and B are on even columns. This shows that every queen from either Y or G is on a different column from every queen in R or B.

As for queens being on the same diagonal, it is clear again that *within* each of the sets Y, G, R and B, the queens are on different diagonals. By symmetry it is sufficient to check if a queen from Y and a queen from G, or a queen from Y and one from R, or a queen from Y and one from B, or a queen from G and one from B are on the same diagonal. As in Case A, we need to check if the slope between two squares, one from one colour and the other from the other colour, can equal +/- 1.

Subcase 1. Y and G: Plugging in the respective row and column indices from Table 1, we find the slope between a square on Y and a square on G is given as $(2(k-i)-2p)/(k-i)$. If this is +1, we find $k-i=2p$, which is impossible as the maximum of k is p while the minimum of i is 1. If the slope is -1, we find $3(k-i)=2p$, which forces p to be congruent to 0 mod 3, violating our hypothesis.

Subcase 2. Y and R. The slope works out to $(-2(i+j)+5)/(2p+1-(i+j))$. Setting this to 1 we find $4=2p+i+j$, which is impossible because p is at least 4. Setting the slope to -1 we find $2p+6=3(i+j)$, which forces p to be congruent to 0 mod 3, violating our hypothesis.

Subcase 3. Y and B. Using the row and column indices from Table 1, we find the slope is given by $(2p-2(i+l)+5)/(2p-(i+l)+1)$. If this is +1, we find $4=i+l$. This is impossible, as i is at least 1, and the condition $p > 3$ shows l is at least 4. If the slope is -1, we find $4p+6=3(i+l)$. Writing $4p$ as $3p+p$, we find p congruent to 0 mod 3, violating our hypothesis.

Subcase 4. G and B. The slope here is $(4p-2(k+l)+5)/(2p-(k+l)+1)$. Setting this to 1 we find $2p+4=k+l$, which is impossible as the maximum of k and l are each p. Setting it to -1 we find $6p+6=3(k+l)$,which is impossible for the same reason.

Finally, *when* $n = 2p + 1$, we place the queens exactly as we did in the corresponding situation in Case A: we place the $2p$ queens in rows 1 through $2p$ exactly as described above, and place the $(2p+1)$-th queen in the square with coordinates $(n, n) = (2p + 1, 2p + 1)$. None of the first $2p$ queens are on the main diagonal given by row index = column index, as is visibly clear from Figure 5 and is easily checked from Table 1. It follows from this and the placement of the $(2p+1)$-th queen that it cannot attack any of the first $2p$ queens.

Remark. An analysis of the proof in Case B shows that the solution for p congruent to 1 mod 3 is also a solution for the case p congruent to 2 mod 3. When $p = 2$, the solutions in Cases A and B are the same, but for higher values of p congruent to 2 mod 3, the two solutions are different.

References

1. Jordan Bell and Brett Stevens, *A survey of known results and research areas for n-queens,* Discrete Mathematics 309 (2009) 1–31

The **DEENS ACADEMY MATH CLUB** began in July 2019 as a small group of tenth graders meeting weekly to discuss mathematics beyond school material and to work on challenging problems. Today it has expanded to double its original membership, and continues to meet weekly using video conferencing. The discussions range from difficult International Mathematics Olympiad material to recreational mathematics. This particular problem was proposed by one of its members Anurag Manoj. He, along with Marina Berchmans, Mohammed Faraaz Ali Khan, Ananya Khare, and Chaitanya Ongole worked on this problem. They solved the problem for small cases of n first (4, 5, 6) and then arrived at the general solution by appealing to analogy. Their mentor Prof. Bharath Sethuraman suggested that they look at slopes of lines joining pairs of queens for their proofs, and they thank him for his guidance.

DIGITAL ROOT OF A PRIME NUMBER

– *Biplab Roy*

An integer $p > 1$ is called a prime if its only positive divisors are 1 and p.

The **digital root** or **seed number** of a natural number in a given number base is the single digit value obtained by an iterative process of summing the digits of the number, on each iteration using the result from the previous iteration to compute a digit sum. The process continues until a single-digit number is reached.

On examining the digital roots of the first 12 prime numbers, we notice something interesting.

There is no prime number less than 40, except 3 itself, whose digital root is 3 or 6 or 9. Can we prove this for all prime numbers? Try it for yourself and if you can't prove the result, turn to page **52.**

Prime Number	Digital Root	Prime Number	Digital Root	Prime Number	Digital Root
2	2	3	3	5	5
7	7	11	2	13	4
17	8	19	1	23	5
29	2	31	4	37	1