Deriving the Golden Ratio using a Scientific Calculator

EYAL N. SHAVIT

In this article, we illustrate the use of a scientific calculator for exploring the Fibonacci sequence and the Golden Ratio. The calculator, equipped with a feature for setting up iterations, enables the process of creating a sequence which rapidly converges to the Golden Ratio.

Introduction

The Fibonacci sequence is one of the best-known mathematical sequences, occurring in nature in a large variety of seemingly unrelated phenomena [1]. The seeds on the spirals of a sunflower and the number of clockwise and anticlockwise spirals in a pinecone exhibit Fibonacci numbers. Each number in the Fibonacci sequence is the sum of the two numbers that precede it. The mathematical equation describing it is

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 1 \text{ with } F_0 = 0 \text{ and } F_1 = 1.$$

The first 12 terms of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.

The ratios of successive terms of the Fibonacci sequence, namely, 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13,. . . approach a value 1.618 (approximately) which is referred to as the Golden Ratio and is represented by the symbol φ. According to Mario Livio [4] the Golden Ratio or φ is "the world's most astonishing number" and can be found not only in natural phenomena but also in a variety of buildings, monuments and works of art.

If we consider three consecutive Fibonacci numbers, $F(i), F(i+1)$ and $F(i+2)$, for sufficiently large values of i, the ratios $F(i+1)/F(i)$ and $F(i+2)/F(i+1)$ are very close to each other and are even identical up to a few places of decimals.

Let

$$\frac{F(i+2)}{F(i+1)} = \frac{F(i+1)}{F(i)} = X \tag{1}$$

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Replacing $F(i+2) = F(i+1) + F(i)$ we get

$$\frac{F(i+1)+F(i)}{F(i+1)} = \frac{F(i+1)}{F(i)} \tag{2}$$

which further becomes

$$1 + \frac{F(i)}{F(i+1)} = \frac{F(i+1)}{F(i)} \tag{3}$$

In terms of X we may rewrite (3) as

$$1 + \frac{1}{X} = X \text{ which simplifies to } X^2 - X - 1 = 0$$

Solving the quadratic equation in X we get $X = \frac{1 \pm \sqrt{5}}{2}$. We ignore the negative value as it is clearly not applicable here. Thus $X = \frac{1+\sqrt{5}}{2} \approx 1.618033989$ (to 9 decimal places) is the Golden ratio or φ.

Here we assumed that the ratios of two successive pairs of terms of the Fibonacci sequence were equal. This only happens *in the limit*. Thus, as the sequence progresses, the ratios get closer and closer to the limiting value, given by φ. This can easily be demonstrated by using a digital computer. Volume 3, issue 1 of At Right Angles (March 2014) included an article [1] which describes the exploration of the Fibonacci sequence and the Golden Ratio using a spreadsheet.

In this article, we shall derive the Golden Ratio using a scientific calculator. Our approach is to solve the equation, $X = 1 + 1/X$ iteratively. The program is very simple, containing essentially one command, which is repeated several times. The calculation converges very rapidly to the value of the Golden Ratio.

In each step of the iteration process, the corresponding numbers of the Fibonacci sequence are displayed on the calculator screen. In order to understand this equation, we at first show the equivalence between the Golden Ratio and its representation as a continued fraction thus defined by the expression

$$\boxed{\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

Because the continued fraction goes on forever, the denominator of the second term is φ itself. The continued fractions can also be defined recursively. Thus the first term is $1 + \frac{1}{1}$ and the second term is $1 + \frac{1}{1+\frac{1}{1}}$. Thus every continued fraction can be written as $1 + \frac{1}{\text{the previous continued fraction}}$.

The value of $1 + \frac{1}{1}$ is 2, the value of $1 + \frac{1}{1+\frac{1}{1}} = 1 + \frac{1}{2} = \frac{3}{2}$.

Proceeding in this manner, we see that the values of the fractions are 2, 3/2, 5/3, 8/5 . . . and so on, which converge to the Golden Ratio φ.

Thus at any stage, we will get a fraction that is the ratio of two consecutive Fibonacci Numbers. In the iteration process we are computing more and more of this continued fraction.

The Recurrence Relation
Here we make use of a recursion relation for calculating the golden ratio, based on the equation $X = 1 + 1/X$. The recurrence relation is given by

$$R_{i+1} = 1 + \frac{1}{R_i} \tag{4}$$

where i denotes the step number of the recursion sequence, with R_i converging to the value of the golden ratio with increasing i. For the initial choice of $R_1 = 1$ the calculated values of R_i as we go down the sequence gives the ratios of successive Fibonacci numbers. Specifically, where $F(i)$ denotes the i-th Fibonacci number, the calculation yields

$$R_i = \frac{F(i+1)}{F(i)} \tag{5}$$

Using the calculator, these ratios can be presented as ratios of natural numbers, thus displaying the Fibonacci sequence. Interestingly, if we start with $R_1 = 1$ we obtain the continued fraction sequence which we have seen in the earlier section.

$$1 + \frac{1}{1}, 1 + \frac{1}{1 + \frac{1}{1}}, 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1}}}, \ldots$$

However, if we start with $R_1 = 5$ (say) we get the continued fraction sequence as

$$1 + \frac{1}{5}, 1 + \frac{1}{1 + \frac{1}{5}}, 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{5}}}, \ldots$$

where the terms lead to the sequence 6/5, 11/6, 17/11,... which converges to the Golden Ratio, more quickly than if $R_1 = 1$.

Programing the calculator. Let us see how to program the iterative procedure on a scientific calculator: The procedure is based on the command **1 + 1/ANS** (ANS is the stored number obtained after pressing the = sign). When we print the above command, each pressing of the = sign repeats the command, with the previous calculated value inserted into ANS. This command provides a natural path for the iterations. Though we have used the Casio fx-82MS calculator, other scientific calculators having the ANS feature may also be used in a similar manner.

The steps to be followed to generate this recursion on the calculator are as follows:

Step 1: Press the number key 1 followed by "=" (this is equivalent to $i = 1$)

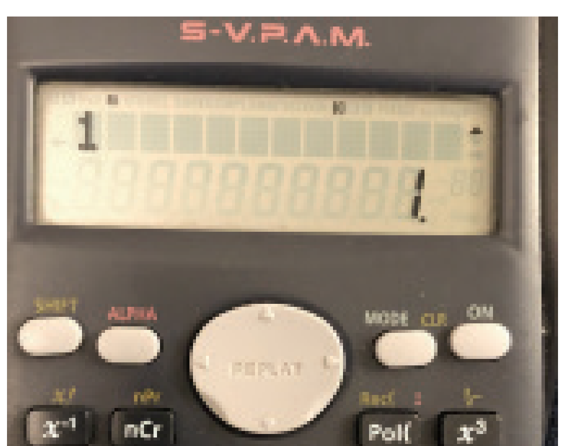

Figure 1. Calculator snapshot for first iteration

Step 2: Type the equation **1 + 1/ANS**, followed by the = sign (this may be repeated as many times as the required number of iterations).

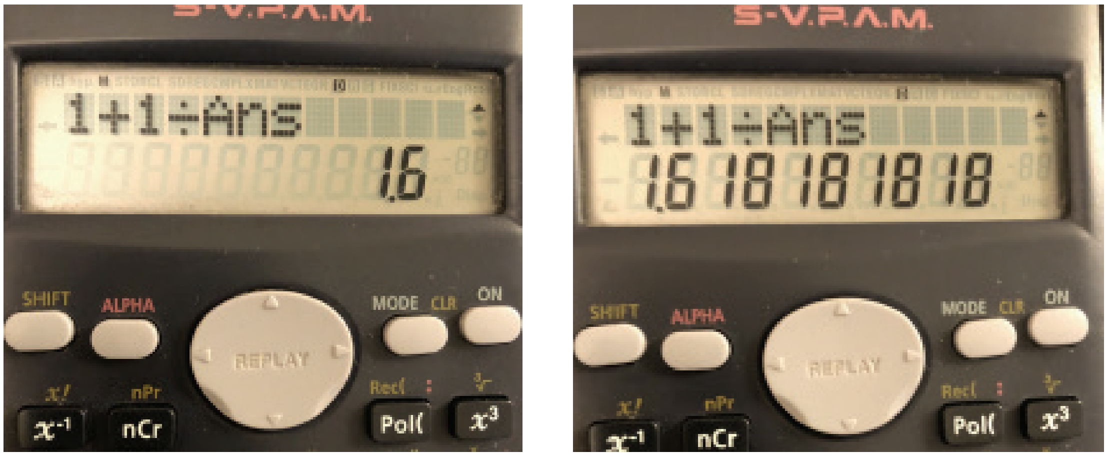

Figure 2. Iteration results for fifth step left and tenth step right.

Each time we press the = key we are asking the calculator to carry out an iteration step of the equation $\boldsymbol{R_i} = \mathbf{1} + \mathbf{1}/\boldsymbol{R_{i-}}$. Pressing **SHIFT** followed by the **ab/c** key converts the calculated number to a fraction, where the numbers in the numerator and denominator are the consecutive terms of the Fibonacci sequence, the ratios of which give the current value of R.

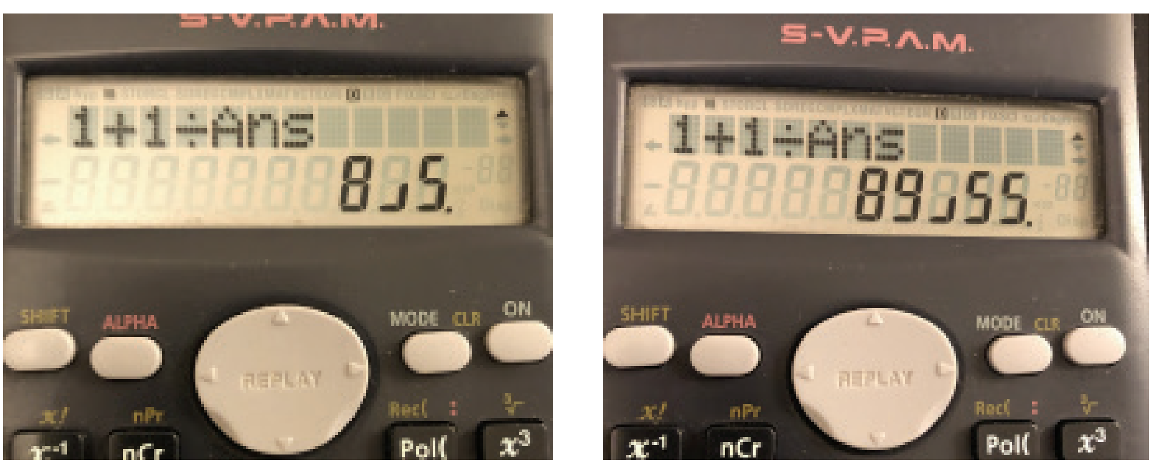

Figure 3. Fifth and tenth iteration values, shown in Figure 2, presented as ratios of Fibonacci numbers.

Results

The result obtained after 23 iterations is the golden ratio, that is, 1.618033989 (up to 9 places of decimals).

Using (4), if $R_1 = 1$ the calculator displays 3/2 in the second iteration, all the way up to 2584/1597 in the 17^{th} iteration.

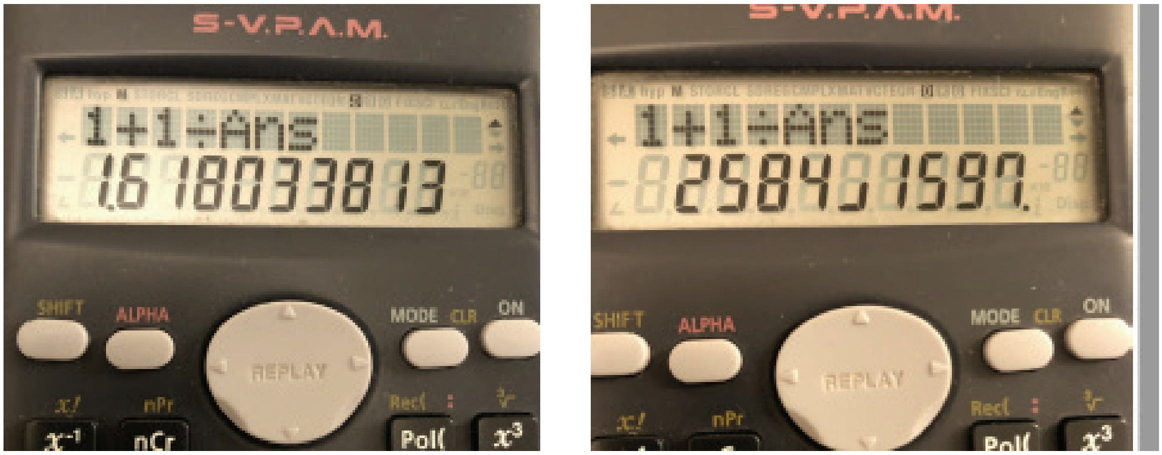

Figure 4. Step 17: On the left iteration value at step 17, on the right this value as presented as ratios of Fibonacci numbers

After the 17th iteration the calculator does not display the Fibonacci numbers.

For values of R_1 not equal to 1 (or to numbers which are given by the ratio of two consecutive Fibonacci terms), the results eventually converge to the above quoted value for the golden ratio, but without the option of displaying the Fibonacci sequence. For $R_1 = 7$ the calculation converges after 24 steps to the value of 1.618033989, the same as for $R_1 = 1$.

The convergence is very robust giving the correct result for R even if R_1 is as large as 10,000, also after 24 steps

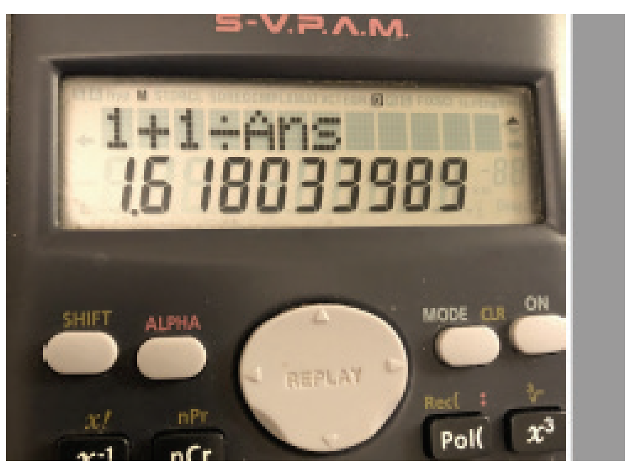

Figure 5. Converged value after 24 steps, with initial value of $R_1 = 10000$

Discussion

While there are software packages available for exploring the Fibonacci sequence and the golden ratio, a simple scientific calculator, if used appropriately, can also lead to deep insights. The recursion feature on the calculator can be effectively utilized to gain insight into the derivation of the golden ratio and other recursive patterns. It is appropriate to mention here, that a similar procedure, although with a larger number of calculator steps, was carried out by Knott [2]. It is also worthwhile to mention the extensive work done by Kissane [3] on the use of scientific calculators in mathematical exploration.

References

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DIGITAL ROOT OF A PRIME NUMBER

– Biplab Roy

Proof: There is no prime number except 3 itself, whose digital root is 3 or 6 or 9. (See page **46** for the observation and the conjecture)

Let $N = a_m.10^m + a_{m-1}\,10^{m-1} + a_{m-2}.10^{m-2} + \ldots + a_1.10 + a_0$ be an arbitrary prime number other than 3, and let $S = a_m + a_{m-1} + a_{m-2} + \ldots + a_1 + a_0$.

It is obvious that, if S is single digit, then S is the digital root of N, otherwise the digital roots of S and N are same.

Also, we know that $3|S \leftrightarrow 3|N$

Since N is a prime number other than 3, 3 is not a divisor of N.

So, 3 is not a divisor of S.

If the digital root of S is 3 or 6 or 9, then 3|S, but this is a contradiction.

So, the digital root of S must not be 3 or 6 or 9.

So, the digital root of N must not be 3 or 6 or 9.

Hence the theorem proved.

References:

[1] Das Gupta, S. (1994) The Story of Prime Number, *Ganita Bharati*, VOL.16, NUMs. 1-4, 37–52.