A Very Special Congruency Problem

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Introduction

The following question appears in Chapter 7 of the Class VII NCERT Mathematics textbook (page 150):

> *Draw a rough sketch of two triangles such that they have five pairs of congruent parts but still the triangles are not congruent.*

Here, 'parts' refers to the three sides and three angles of the triangle; 'congruent parts' means that the corresponding parts of the two triangles are identically equal to each other. The questions that quickly come to mind are: "Why a rough sketch? Why not the exact figure?" "Is it possible to draw such pairs of triangles exactly?" We will find the answers to these questions in this article and prove some related results.

The four congruence theorems

Students are familiar with four congruence criteria, i.e., SSS, SAS, ASA (equivalently, AAS), and RHS. So if three parts of one triangle are equal to three parts of another triangle, as per these criteria, then the triangles are congruent. But if someone asks, *Is it possible that two triangles have five pairs of congruent parts and yet are not congruent?* the students' impulsive answer is, *NO, it is not possible*. At that moment if you tell them *YES, it is possible*, they just cannot believe it and they ask the immediate question, *How?* Well, let's investigate the reason behind it.

Keywords: NCERT, congruent, congruence criteria, SAS, SSS, ASA, AAS, RHS

Analysis

We will describe how one can construct such pairs of triangles. If we make three pairs of sides equal, then the triangles will be congruent. So in the list of five congruent pairs, three pairs of congruent sides cannot be included. Rather, it must be two pairs of congruent sides and three pairs of congruent angles. In this case, the triangles will be similar but not necessarily congruent. Now, if the equal pairs of sides correspond to equal pairs of angles, then the triangles will be congruent to each other, by SAS or ASA. As we do not want congruence, we must ensure that corresponding sides of the triangles are not equal to each other. So we have to make two pairs of non-corresponding sides of the triangles equal in length. Then it is possible to create such constructions. The following two figures show one such instance.

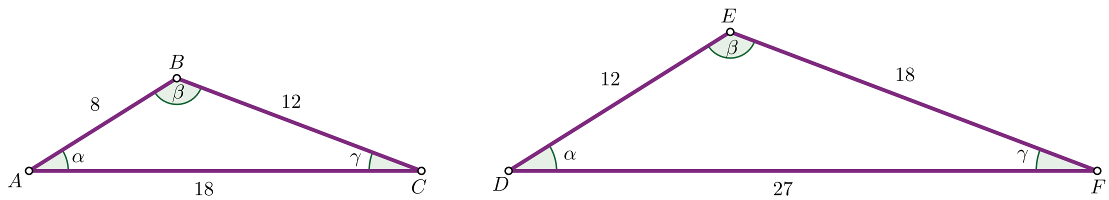

Figure 1.

From the above figures, we see that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{2}{3}.$$

So the triangles are similar but non-congruent.

Again in both triangles, two pairs of sides are equal, i.e., $BC = DE$ and $AC = EF$, but they are non-corresponding sides.

Now we will characterize all such pairs of triangles.

Let the sides of the first triangle be a, b, c, and let the sides of the second triangle be b, c, d. (Observe that there are two pairs of equal sides.) Since the triangles must be similar to each other, with a, b, c corresponding to b, c, d respectively, we must have:

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}. \tag{1}$$

Now we must choose a, b, c and b, c, d in such a way that they must form genuine triangles. Let

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k, \tag{2}$$

where $k > 0$ but $k \neq 1$ (else the triangles are equilateral).

So clearly, $c = dk$, $b = ck = dk^2$ and $a = bk = ck^2 = dk^3$. The sides of the two triangles are

$$(a, b, c) = (dk^3, dk^2, dk) = (ck^2, ck, c), \tag{3}$$

$$(b, c, d) = (dk^2, dk, d). \tag{4}$$

We know that three line segments form a triangle if (and only if) the sum of the lengths of any two line segments is greater than the third. So $a + b > c$, $b + c > a$ and $c + a > b$ must hold simultaneously, with similar conditions for b, c, d. Both lead to these inequalities:

$$k^2 + k > 1, \tag{5}$$

$$k + 1 > k^2, \tag{6}$$

$$k^2 + 1 > k. \tag{7}$$

From (5) we have $k^2 + k > 1$, hence

$$k^2 + k - 1 > 0, \quad \therefore \left(k + \frac{\sqrt{5}+1}{2}\right)\left(k - \frac{\sqrt{5}-1}{2}\right) > 0.$$

Since $k > 0$ we must have

$$k > \frac{\sqrt{5}-1}{2}. \tag{8}$$

From (6) we have, $k + 1 > k^2$, hence

$$k^2 - k - 1 < 0, \quad \therefore \left(k - \frac{\sqrt{5}+1}{2}\right)\left(k - \frac{1-\sqrt{5}}{2}\right) < 0.$$

Since $k > 0$ we must have

$$k < \frac{\sqrt{5}+1}{2}. \tag{9}$$

From (7) we have, $k^2 + 1 > k$, hence $k^2 - k + 1 > 0$, but this is always true for all real k, as the discriminant of the corresponding quadratic $k^2 - k + 1$ is $(-1)^2 - 4 = -3 < 0$.

From (8) and (9) we get:

$$\frac{\sqrt{5}-1}{2} < k < \frac{\sqrt{5}+1}{2}. \tag{10}$$

But we know $k \neq 1$. So we have:

$$\frac{1}{\phi} < k < 1 \quad \text{and} \quad 1 < k < \phi, \tag{11}$$

where ϕ is the *golden ratio*:

$$\phi = \frac{\sqrt{5}+1}{2}.$$

By choosing different values of k between these intervals, we can generate infinitely many such pairs of triangles.

We know that the golden ratio $\phi = \frac{\sqrt{5}+1}{2} = 1.61803398874$, so the second interval $1 < k < \phi$ yields $1 < k < 1.618\ldots$.

To construct pairs of triangles having the required property and the added property that the lengths of the sides of the two triangles are integers, we make use of the above analysis and the following sequence,

$$\left\{\frac{n+1}{n}\right\},$$

which steadily decreases and converges to 1:

$$1 < \cdots\cdots < \frac{6}{5} < \frac{5}{4} < \frac{4}{3} < \frac{3}{2} < \frac{2}{1}. \tag{12}$$

Except for the first term, $\frac{2}{1}$, all other terms of the sequence lie in the second interval, and the first value of the sequence which satisfies (11) is $\frac{3}{2}$.

Choosing $k = \dfrac{3}{2}$, we get the side lengths as

$$(a, b, c) = (dk^3, dk^2, dk) = \left(\frac{27d}{8}, \frac{9d}{4}, \frac{3d}{2}\right),$$
$$(b, c, d) = (dk^2, dk, d) = \left(\frac{9d}{4}, \frac{3d}{2}, d\right).$$

To ensure that all the lengths are integers, the least value of $d = 8$. This gives a pair of triangles whose side lengths are $(27, 18, 12)$ units and $(18, 12, 8)$ units respectively. These two triangles are shown in Figure 1.

If we choose $k = \frac{4}{3}$ and $d = 27$, we get two triangles whose side lengths are $(64, 48, 36)$ units and $(48, 36, 27)$ units respectively. These two triangles also satisfy the required condition.

Now in general, if we choose $k = \frac{n+1}{n}$ where $n \geq 2$ is an integer, we get,

$$(a, b, c) = \left(\frac{(n+1)^3}{n^3}d,\ \frac{(n+1)^2}{n^2}d,\ \frac{(n+1)}{n}d\right),$$
$$(b, c, d) = \left(\frac{(n+1)^2}{n^2}d,\ \frac{(n+1)}{n}d,\ d\right).$$

If we take $d = n^3$, then we get two integer-sided triangles whose side lengths are given by

$$(a, b, c) = \left((n+1)^3,\ n(n+1)^2,\ n^2(n+1)\right),$$
$$(b, c, d) = \left(n(n+1)^2,\ n^2(n+1),\ n^3\right).$$

Notice that among these six sides, the smallest side is n^3, and the biggest side is $(n+1)^3$.

So from the above results, we can state the following theorem,

Theorem. *Let $n \geq 2$ be a positive integer. Then there exists a pair of integer-sided triangles with five pairs of congruent parts, such that the triangles themselves are not congruent to each other, the shortest side of the smaller triangle being n^3, and the longest side of the larger triangle being $(n+1)^3$.*

That means that for every two consecutive integer cubes, there exists a pair of integer-sided triangles with five pairs of congruent parts, such that the triangles themselves are not congruent to each other, the shortest side of the smaller triangle being the smaller cube, and the longest side of the larger triangle being the larger cube.

Note that if we choose the first interval $\frac{1}{\phi} < k < 1$, we arrive at the same conclusion. In this case, we could choose the sequence $\left\{\frac{n}{n+1}\right\}$ and the starting value $k = \frac{2}{3}$, instead of $k = \frac{3}{2}$.

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