Features

On some Geometric Constructions in the Sulvasutras from a Pedagogical Perspective – I

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Methodical geometric ideas flourished in ancient India in the context of the engagement with construction of *vedi*s (altars, or platforms) and *agni*s (fireplaces) for performance of *yajna*s (fire worship), which are a hallmark of the Vedic civilization. The *sulvasutras*[1] are compositions giving an exposition of the procedure for erecting the ritual structures involved, which also incorporate along the way descriptions of various geometric principles and constructions. It may be worth recalling here, without going into the details, that the structures involved were of large size (extending to several meters on ground, not amenable to hand drawing) and in intricate shapes representing birds, tortoise, etc. (see [1], [3], [4] or [6] for details), which seems to have generated interest in geometric theory.

The geometry of sulvasutras has many commonalities with Euclidean geometry, and in particular we may recall here that the Pythagoras theorem is stated explicitly in the sulvasutras; there are four sulvasutras that are renowned for their mathematical content and all four of them include the theorem. One of the themes that pervades the sulvasutra

1 In this article we will not use diacritical marks, as commonly used in technical literature in the subject to indicate pronunciation of Sanskrit words, except for certain special words occurring in isolation. A pronunciation guide for the words used is included in a Glossary.

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geometry is constructing new figures *with the same area* as a given one, or equivalently turning the given shape into other shapes, retaining the same area. Consideration of such an equivalence of geometric figures brings to mind the role played by congruence or similarity in Euclidean geometry, while on the other hand it may also be construed as the issue of constructing desired shapes with a given area.

Our aim in this article is to discuss various instances along this theme, occurring in the sulvasutras. We believe that this would have pedagogical benefits, as the material nicely complements geometry in schools (at 6th to 8th standards) and an exposure to the different perspective involved could enhance the students' interest, and their ability, in getting a better grasp of geometry. The article will be in two parts; in Part I we focus on constructions of basic rectilinear shapes (namely those with straight edges), and in Part II, together with some more general developments concerning these, construction of semicircles and circles with given areas, and certain broader mathematical issues related to the constructions will be discussed.

The instructions in the sulvasutras presume a certain overall familiarity in various respects, on the part of one receiving the instruction. Such a familiarity would be acquired in their time through oral communication, and the compositions were meant to supplement it, as a supporting device. Thus what is put down was mostly intended only to aid recollection, and not aimed at giving a detailed or precise description. A literal translation of the text involved would therefore not be very useful to a modern reader, and a degree of paraphrasing is needed to convey what is meant. It will nevertheless be our endeavour here to stay close to the straightforward meaning of the text[2], keeping the paraphrasing to a minimum, in contrast to the tendency in some of the writing in the area to resort to generous (and sometimes unjustifiable) paraphrasing. This we believe would give the reader a better insight into how the ancients thought of the issues involved.

Typically, the desired transformations were achieved through geometrical procedures, though we do find, as will be seen in Part II of the article, an exception to this at an advanced level. While some of the constructions are based on elementary considerations, or what may be termed 'visual geometry,' others are based on the Pythagoras theorem, and still others (discussed in Part II) involve some arithmetic as well. In developing the theme here we shall begin with the simpler and more basic ideas and proceed gradually, through ascendant sections, to more complex ones, and not adhere to the sequence of their occurrence in the sulvasutras.[3]

As mentioned above there are four sulvasutras known for their mathematical, primarily geometrical, content; Baudhayana sulvasutra, Apastamba sulvasutra, Manava sulvasutra and Katyayana sulvasutra. Of these, Baudhayana is the earliest (ca. 800 BCE) and Katyayana the latest (ca. 200 BCE)[4]; the reader may consult the references cited at the end (of this part) for various details concerning sulvasutras, especially the mathematical aspects. We shall mostly refer to the Baudhayana sulvasutra, which is the most comprehensive and systematic in respect of exposition, even though it is the oldest. There are, however, certain ideas in the other sulvasutras, related to the theme at hand, that are not found in Baudhayana, which also we shall discuss; we do not aim at being comprehensive, however, in covering all instances along the theme, but endeavour to convey the variety and essence involved. References to the original sutras are included for the benefit of the interested reader; for sutras from Baudhayana Sulvasutra the sutra reference is marked with BSS, and similar abbreviations will be used for others, which will be clear from the context.

2 It may be mentioned here that while we have greatly benefited from translations available in literature, the translations presented here are ultimately our own.

3 The considerations that may have gone into the organization of the contents of the sulvasutras, with substantial variations among the individual sulvasutras, are not quite clear, and may provide a worthy topic for exploration by itself.

4 The dates assigned to the sulvasutras are estimates based on rather general considerations, with no specific evidence, and may involve a large error margin.

I. Elementary constructions

In this section we discuss constructions that are based on elementary principles, or what one may call visual geometry.

I.1. Constructing isosceles trapezia. Trapezia are a frequent occurrence in the Sulvasutras, mainly as shapes of various vedis. They are invariably isosceles (symmetrical), symmetric about the line joining the midpoints of the two parallel sides; the line of symmetry is set along the east-west direction, with the face of the trapezium (the smaller of the parallel sides) towards the east. There is no name given to the figure; at many places they turn up as the end product of a construction described (for the shape of the desired vedi), and when a reference to the figure is called for, it is made through a description which goes somewhat like "quadrilateral pointed (*aṇimat*) on one side."

The sutra BSS 2.6 (1.55)[5], from Baudhayana sulvasutra, instructs on how to produce an isosceles trapezium starting with a square (or rectangle), which may be stated as follows:

> When a square is desired to be converted to one that is pointed on one side (in the form of an isosceles trapezium, having the same area as the initial square) keeping a transverse segment of the size desired for the shorter side (at one end), the remaining (rectangular) part is to be divided along its diagonal and the (triangular) excess part is to be inverted and adjoined on the other side.

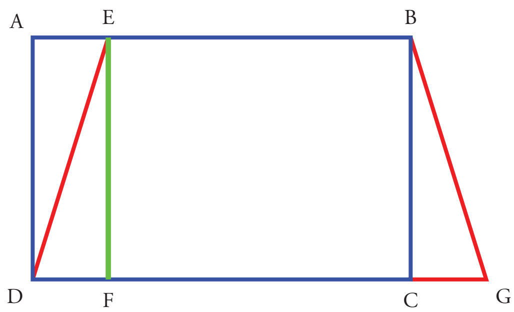

Figure 1. Rectangle ABCD converted to trapezium EBGD

Though the construction is described starting with a square, the same applies just as well to any rectangle; Figure 1 illustrates the procedure involved for a general rectangle; here ABCD is the initial rectangle and EB is the desired reduced size taken as the 'transverse;' the latter term, corresponding to *tiryaṅmānī* in the original sutra, stands for line segments cutting across the verticals (which were set in the east-west direction in their context); EBGD is the isosceles trapezium produced following the procedure, by moving the triangle ADE to the other side.

I.2. Construction of a rhombus. Some vedis are in the shape of a rhombus, namely a quadrilateral with four equal sides, the diagonals being (typically) unequal; in this case the longer diagonal is set along the east-west direction.

BSS 2.8 (1.57) gives a procedure for constructing a rhombus, which goes as follows:

> When a rhombus of the size of a (given) square is desired, produce a rectangle with area twice that of the square and join the midpoints of the sides of the rectangle.

5 The number of the sutra as in [6] is given first, and is followed parenthetically by the numbering adopted in some of the older sources, including [3] and [4].

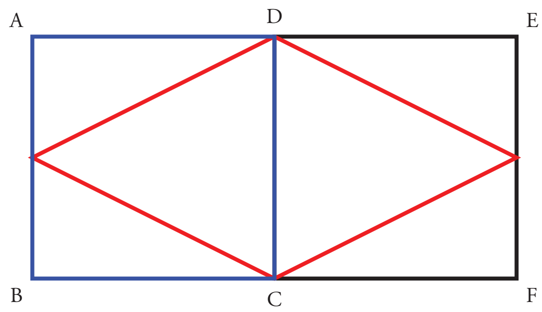

Figure 2. Constructing a rhombus with the same area as a square

The construction is illustrated in Figure 2. To the given square ABCD is adjoined an identical square CDEF, and then the midpoints of the four sides of the rectangle ABFE are joined cyclically. Where we have said "join the midpoints" the original sutra asks for poles to be erected at those midpoints, ropes to be stretched between the poles (tied pairwise, cyclically), and taking away the part outside the ropes – which is a description in a practical context. Though it has not been specified in the sutra, a rectangle with double the desired area would presumably be produced by putting two squares with the given area alongside; as per their convention which for convenience we have chosen not to adhere to in Figure 2, the additional square would be placed towards the south, so that the longer diagonal will be along the east-west direction; this specific detail in the procedure is seen adopted in the construction of a rhombus of a specific size described in Manava sulvasutra (MSS 15.4 (10.3.6.4)).

I.3. Converting a square to a rectangle. BSS 2.3 (1.52) describes the following construction to turn a square into a rectangle.

> Wishing to turn a square into a rectangle, cut it diagonally, then divide one part again and adjoin the two halves along the sides (of the original square).

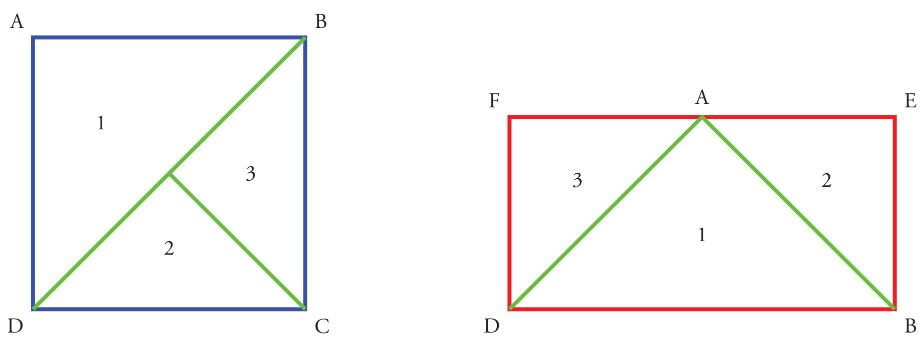

Figure 3. The square ABCD is rearranged into the rectangle FEBD

This construction should be clear from the illustration in Figure 3; the square ABCD is divided into parts and the pieces are reassembled according to the prescription, as indicated by the numbering. The procedure is evidently for producing specifically a rectangle whose sides are in the ratio 2 : 1. BSS 2.4 (1.53) purports to give a construction of a rectangle with more general side length, from a given square. However the description is rather too vague (commented as 'defective' in [6], page 79). The construction described by Apastamba (ASS 3.1 (3.1)) for the same purpose is a bit more specific, mentioning *any desired size* for the side of the rectangle to be produced, but on the whole the purported construction is still not quite clear from the description. Commentators Dwarakanatha of Baudhayana sulvasutra and Sundararaja of Apastamba sulvasutra give a construction in this respect, by way of interpretation of the sutras as above, which is as illustrated in Figure 4.

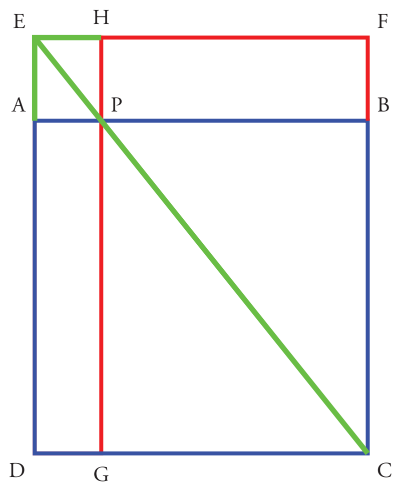

Figure 4. The square ABCD transformed to the rectangle HFCG

Given the square ABCD and a prescribed size for the side, given here by the extended segment DE, one joins E to C and plots the point P of intersection of AB and EC. The points G and H are determined on the vertical (line through P parallel to DE), with G on the segment DC and H at the level of E. A rectangle HFCG is now drawn with GH and GC as the two sides; its area may be readily seen to be equal to that of the given square ABCD. While the construction indeed seems worth noting here in the overall context, it is unclear to what extent the interpretation of the commentators (coming over a thousand years later) may be associated with what the sutrakaras would have had in mind.

Remark I.1. BSS 1.9 (1.45) states:

> The diagonal of a square makes twice the area.

In the sulvasutras, when a line segment is said to 'make' a certain area, it is meant that the latter is the area of the square over the segment. Thus the above statement means that the square over the diagonal has twice the area as the original square. In the context of this property the diagonal of a square acquired a special name, *dvikaraṇī* (side that doubles).

The statement in Remark I.1 is clearly a special case of the Pythagoras theorem when the two sides of the right angled triangle are equal; on the other hand it is more elementary compared to the general case, and visually evident. It has been stated by Baudhayana separately in the form as above, before going to the general statement of the theorem, appearing in BSS 1.12 (1.48), discussed in § 2 below. In particular, it may be observed that the validity of this special case of the theorem is equivalent to the equality of the areas of the rectangle and square in §1.3 as above, as two of those rectangles can be joined along the longer side to get a square with sides equal to the diagonal of the original square.

I.4. Construction of an isosceles triangle from a square. An isosceles triangle of the size of a given square can evidently be constructed in a way similar to that of the construction of a rhombus, described in §1.2, by producing a rectangle with twice the area of the given square and joining the vertices at the base to the midpoint of the opposite side. BSS 2.7 (1.56) describes a specific construction of an isosceles triangle in this respect, in which a *square* is instructed to be chosen in place of the rectangle:

> To construct an isosceles triangle of the size of a (given) square, form a square with twice the area as the given square and join the midpoint of one of the sides to the vertices of the side opposite to it.

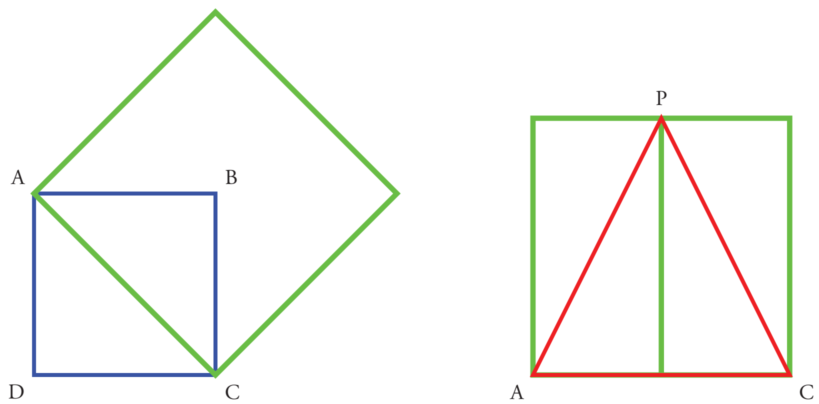

Figure 5. An isosceles triangle with the same area as a given square

Here the sutra specifically alludes to a *square* of twice the area being formed at the intermediate stage, using the term *samacaturasra*; geometrically, this would be mandated, for instance, if the altitude of the isosceles triangle to be produced is required to be of the same size as the base. The procedure is illustrated in Figure 5, where an isosceles triangle PAC is produced with the same area as the given square ABCD. We recall from Remark I.1 that a square with twice the area of the given square can be formed geometrically, taking the diagonal of the given square for its side, thus fulfilling the first step above. The second step, illustrated in the second part of Figure 5, then produces an isosceles triangle with area equal to that of the original square ABCD. As in the comment in §1.2, in the present instance also, joining of the midpoint to the two vertices is described in the sutra in the practical context, mentioning poles and ropes; moreover the midpoint is taken on the side towards the east, the square chosen being arranged along the cardinal directions; the isosceles triangles involved in the ritual constructions had to be pointing to the east.

II. Constructions based on the Pythagoras theorem

To begin with we recall here that the Pythagoras theorem is stated in all the four sulvasutras, mentioned earlier, and that the early ones among them would be considerably prior to Pythagoras (ca. 570 - ca. 495 BCE).[6] The statement in Baudhayana sulvasutra, in BSS 1.12 (1.48), may be translated as

> The diagonal of a rectangle makes both of what the flank and the transverse sides make separately.

The reader would recognise that with the meaning of 'make' discussed in Remark I.1, this is equivalent to the Pythagoras theorem in its general form, stated with respect to rectangles, in place of the right angled triangles. The terms 'flank' and 'transverse' are adopted in the above translation as they closely correspond in their meaning to the original terms *pārśvamānī* and *tiryaṅmānī*, respectively, in the sutra, though in paraphrasing the sutra one may simply refer to the length and breadth of the rectangle; the term used for rectangle is *dīrghacaturasra* and for the diagonal it is *akṣṇayārajju*. The statements in the Apastamba and Katyayana sulvasutras are similar, with minor variations that we need not go into. The Manava sulvasutra (MSS 10.10 (10.3.1.10)) presents the theorem in a different form, which may be worth recalling, for its distinct 'algorithmic' form.

> Product of the width with the width and stretch with the stretch, when added and taken square root of, is the diagonal – this is known.

6 The Pythagoras theorem was also known in the Babylonian civilization, close to 2000 BCE, and perhaps also to the ancient Egyptians.

'Width' and 'stretch' here stand for the two sides of a rectangle ('*āyāma*' and '*vistāra*,' respectively, in the original) whose diagonal is the object to be determined. Incidentally, *karṇa* is the word used in the sutra for the diagonal; it was carried through in later Sanskrit literature, and is now in usage in various Indian languages.

The theorem is used in various transformation problems for figures that we describe through this section.

II.1. Combining two squares into one square. BSS 2.1 (1.50) describes the following procedure for putting together two squares into one:

> To combine two squares of different sizes into one square, plot the rectangle in the bigger square, through the rising point (in the bigger square, when the two are placed side by side); the diagonal of the rectangle becomes the side for the joint square.[7]

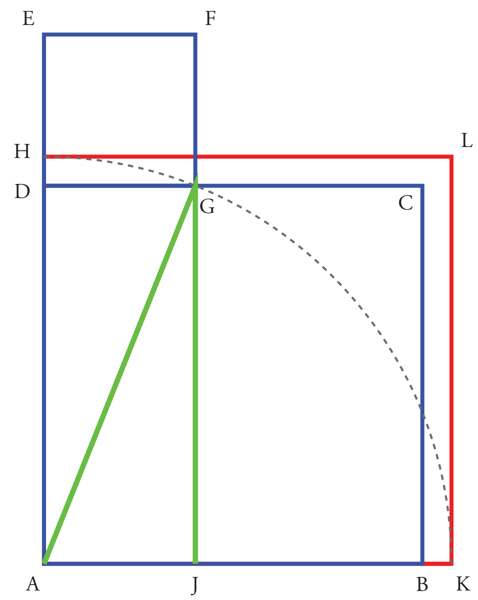

Figure 6. Two squares ABCD and EFGD combined into one square AKLH

The construction is illustrated by Figure 6.[8] Given the squares ABCD and DEFG placed alongside as set in the figure, the rectangle ADGJ through the 'rising point' G is plotted. The side AH of the desired square is taken to be of the size of the diagonal AG of the rectangle ADGJ. As the sides of the latter equal the sides of the two given squares, AH has the desired property, by a straightforward application of the Pythagoras theorem.

II.2. Converting the difference of two squares to a square. BSS 2.2 (1.51) gives the following prescription for converting the difference of two squares, of different sizes, into a square.

> When a square is to be taken away from a square, plot a rectangle in the bigger square along the side of the smaller square; drop the flank side of the rectangle on its other side as the diagonal; the segment which it cuts off is the answer.

7 Regarding this translation we would like to mention that, as there are some apparently technical terms involved in the statement, that have in fact been a subject of discussion going back to Thibaut [7], in this instance we have relied to a larger extent on the received wisdom on what the sutra means, than on our own linguistic resources.

8 In illustrating the construction, often the squares are drawn in an overlapping fashion - in our view it would however be more appropriate, given the spirit of what is involved, to draw them side by side as shown here.

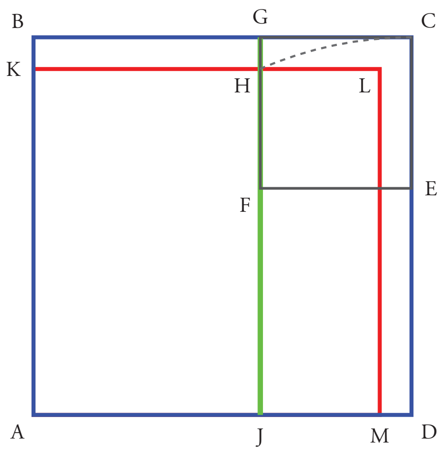

Figure 7. Difference of squares ABCD and CEFG is turned into a square AKLM

The construction is illustrated by Figure 7. Consider a square ABCD, and a square CEFG to be taken away from it, as seen in the figure. Then the rectangle CDJG is plotted with the point J on AD so that JD equals GC. The flank side CD of CDJG is dropped on the opposite side JG, through an arc drawn with centre at D and radius DC, and the point H where it meets FG is marked. A square AKLM is drawn with side equal to JH. This gives a square with the desired property, again as an application of the Pythagoras theorem; here D, J and H form a right angled triangle in which the hypotenuse DH has length equal to the side of the larger square and JD has length equal to that of the smaller square, so the square on the side JH, and equivalently on AK, has area equal to the difference of the two squares.

II.3. Converting a rectangle to a square. BSS 2.5 (1.54) describes the following procedure for converting a rectangle into a square; the same is also described in KSS 3.2.

> When a rectangle is desired to be converted to a square, taking out a square over the transverse side, divide the remaining part of the rectangle into two, and move one part to the flank side; that (the figure formed) can be supplemented by a square to form a square, and how to rectify that (to get a square as desired) has been explained earlier.

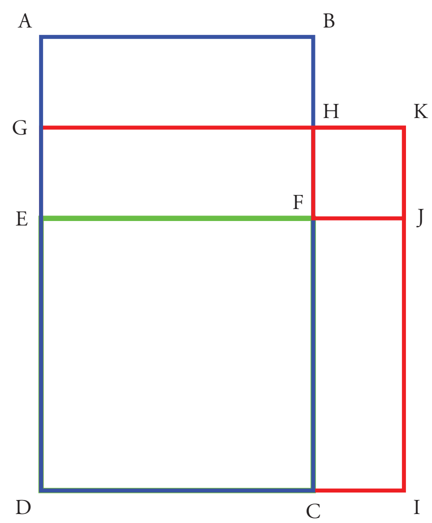

Figure 8. Squaring a rectangle

The construction is illustrated in Figure 8.[9] Given the rectangle ABCD, a square CDEF is marked out at one end and the remaining rectangle is divided in the middle along the shorter side, by joining the midpoints G and H of AE and BF respectively. The rectangle ABHG is moved to the other side, relocating it as IJFC. The rectangle is now transformed to a figure which is readily seen to be a difference of two squares, viz. of GKID and HKJF. The instruction at this point is to follow the procedure described earlier (in §2.2 in the present exposition) to convert the difference into a square.[10]

II.4. Converting an isosceles triangle to a square. Katyayana makes use of the process of conversion of rectangle into square, in describing a procedure to turn an isosceles triangle into a square. KSS 4.5 (4.7) has the following instruction in this respect.

> When an isosceles triangle is sought to be turned to a square, divide it along the bisector, invert one of the parts and place it along the other to get a rectangle, and convert the rectangle to a square.

It should be clear how an isosceles triangle would be turned into a rectangle by dividing along the altitude and reassembling along the slanted sides. After obtaining a rectangle the procedure as in §2.3 is to be followed. Thus altogether we have here a three-tier process of construction!

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9 The sutra assumes the longer side to be along the east-west direction (vertical in our representation) consistent with a general convention seen in the sulvasutra constructions - of course from a geometric point of view this involves no loss of generality.

10 This is perhaps the oldest instance in which in a mathematical discourse an appeal is explicitly made to a result described earlier, that is now a common practice in mathematics.

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Glossary of names and terms

As in text	As in technical literature	In Devanagari script
Apastamba	Āpastamba	आपस्तम्ब
Baudhayana	Baudhāyana	बौधायन
Circle	Maṇḍala	मण्डल
Diagonal (1)	Akṣaṇayā/ Akṣaṇayārajju	अक्ष्णया/अक्ष्णयारज्जु
Diagonal (2)	Karṇa	कर्ण
Flank (longitudinal) side	Pārśvamānī	पार्श्वमानी
Isosceles triangle	Prauga	प्रउग
Katyayana	Kātyāyana	कात्यायन
Manava	Mānava	मानव
Pointed	Aṇimat	अणिमत्
Puruṣa (height of man with uplifted arms)	Puruṣa	पुरुष
Quadrilateral	Caturasra	चतुरस्र
Rectangle	Dīrghacaturasra	दीर्घचतुरस्र
Rhombus	Ubhayataḥ prauga	उभयतः प्रउग
Rope or cord	Rajju, Śulva/Śulba	रज्जु, शुल्व/शुल्ब
Semicircle	Ardhamaṇḍala	अर्धमण्डल
Square (1)	Caturasra	चतुरस्र
Square (2)	Samacaturasra	समचतुरस्र
Stretch	Vistāra	विस्तार
Sulvasutra	Śulva-sūtra/ Śulba-sūtra	शुल्बसूत्र
Sutra (statement in aphoristic style)	Sūtra	सूत्र
Sutrakara (composer of sutras)	Sūtrakāra	सूत्रकार
Transverse (lateral) side	Tiryaṅmānī	तिर्यङ्मानी
Width	Āyāma	आयाम
Yajamana (master of ceremony)	Yajamāna	यजमान
Yajna (fire worship/ritual)	Yajña	यज्ञ