Secondary

Problem Corner

A Problem concerning Prime Numbers

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In this paper, I explore the following problem:

Problem. Find instances where a product of distinct prime numbers is an integer multiple of the sum of the same prime numbers. Find as many such instances as possible.

Here is an example of such a situation:

$$\frac{2 \times 3 \times 5}{2+3+5} = 3, \quad \text{an integer.}$$

So the problem is to find prime numbers $p, q, r, \ldots$, with $p < q < r < \cdots$, such that

$$\frac{pqr\cdots}{p+q+r+\cdots} = \text{an integer.} \qquad (1)$$

Let k denote the number of primes in $\{p, q, r, \ldots\}$. The case $k = 1$ is trivial, as the fraction simplifies to 1. In this article, we study the cases $k = 2$, $k = 3$, and $k = 4$ (the last case only partially).

The case k = 2. Here the problem is to find primes p, q (with $p < q$) such that pq is an integer multiple of $p + q$. We shall show that it is not possible to find any such pair.

To start with, note that if both p, q are odd, then pq is odd whereas $p + q$ is even, and an even number obviously cannot be a divisor of an odd number. Hence this situation cannot occur.

The case left to be considered is when the smaller prime is 2. Suppose that $p = 2$. Then the condition is that $2q$ is an integer multiple of $q + 2$, with $q > 2$. However, if $q > 2$, then

$$q + 2 < 2q < 2(q + 2).$$

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So it is not possible for $2q$ to be an integer multiple of $q+2$ if $q>2$. Hence this situation too cannot occur.

This means that there are no solutions when $k=2$.

The case k = 3. Here the problem is to find primes p, q, r (with $p<q<r$) such that pqr is an integer multiple of $p+q+r$. We have seen one such instance, above. The analysis turns out to be quite rich in this case. We start by considering the cases $p=2$ and $p=3$.

The case $p=2$. Suppose that $p=2$. We need to find odd primes q, r, with $2<q<r$, such that

$$2qr = n(2+q+r), \quad \text{for some positive integer } n. \tag{2}$$

The prime factorisation of the quantity on the left side of (2) is $2\times q\times r$. Hence $2+q+r$ is one of the following:

$$2,\ q,\ r,\ 2q,\ qr,\ 2r,\ 2qr.$$

All but two of these possibilities can be immediately eliminated. For example, suppose that $2+q+r=qr$. This may be written as

$$qr-q-r=2,$$
$$\therefore\ (q-1)(r-1)=3.$$

The last equality is not possible, since $r\geq 5$. Hence it is not possible that $2+q+r=qr$.

Continuing, we find that the only possibilities left are $2+q+r=2q$ and $2+q+r=2r$. We consider both these in turn.

- If $2+q+r=2q$, then $2+r=q$, or $r=q-2$. But this is not possible, since we have supposed that $r>q$.
- If $2+q+r=2r$, then $2r=2+q+r$, so $2+q=r$. **This means that q, r are a pair of twin primes!** The last case has yielded a very nice conclusion!

We see that the instance $(2,3,5)$ is not an isolated one (note that $3, 5$ are a pair of twin primes); $(2,q,r)$ is a solution for any pair of twin primes q, r. We thus have the following instances:

- $(2,5,7)$, with $2\times 5\times 7 = 5\times(2+5+7)$;
- $(2,11,13)$, with $2\times 11\times 13 = 11\times(2+11+13)$;
- $(2,17,19)$, with $2\times 17\times 19 = 17\times(2+17+19)$;
- $(2,29,31)$, with $2\times 29\times 31 = 29\times(2+29+31)$; and so on.

The case $p=3$. Suppose that $p=3$. We need to find odd primes q, r, with $3<q<r$, such that

$$3qr = n(3+q+r), \quad \text{for some positive integer } n. \tag{3}$$

The prime factorisation of the quantity on the left side of (3) is $3\times q\times r$. Hence $3+q+r$ is one of the following:

$$3,\ q,\ r,\ 3q,\ qr,\ 3r,\ 3qr.$$

All but two of these possibilities can be immediately eliminated. For example, suppose that $3+q+r=qr$. This may be written as

$$qr-q-r=3,$$
$$\therefore\ (q-1)(r-1)=4.$$

The last equality is not possible, since $r \geq 7$. Hence it is not possible that $3 + q + r = qr$.

Continuing, we find that the only possibilities left are $3 + q + r = 3q$ and $3 + q + r = 3r$. We consider both these in turn.

- If $3 + q + r = 3r$, then $3 + q = 2r$. Now we have $r \geq q + 2$, so we get $3 + q \geq 2q + 4$, or $q \leq -1$, which is absurd. Hence this possibility does not work out.
- If $3 + q + r = 3q$, then $3 + r = 2q$. This means that q is the arithmetic mean of 3 and r. **Otherwise expressed, the primes 3, q, r form an arithmetic progression!** Once again, the last case has yielded a very nice conclusion.

  To list such instances, we need to list primes q such that $2q - 3$ is also prime. There are many primes with this property, for example:

$$5,\ 7,\ 11,\ 13,\ 17,\ 23,\ 31,\ \ldots,$$

  resulting in the following triples which satisfy the given condition:

$$(3,5,7),\quad (3,7,11),\quad (3,11,19),\quad (3,13,23),\quad (3,17,31),\quad \ldots.$$

The case $p = 5$. Suppose that $p = 5$. We need to find odd primes q, r, with $5 < q < r$, such that

$$5qr = n(5 + q + r), \qquad \text{for some positive integer } n. \tag{4}$$

The prime factorisation of the quantity on the left side of (4) is $5 \times q \times r$. Hence $5 + q + r$ is one of the following:

$$5,\ q,\ r,\ 5q,\ qr,\ 5r,\ 5qr.$$

Arguing as earlier (we leave out the details; please try to fill in the details yourself), we find that only one possibility works out: $5 + q + r = 5q$. This leads to the following:

$$5q = 5 + q + r, \quad \therefore \quad r = 4q - 5. \tag{5}$$

So we must look for odd primes $q > 5$ such that $4q - 5$ too is prime. Here are some primes that satisfy this condition:

$$7,\ 13,\ 19,\ 43,\ 61,\ 67,\ 79,\ 97,\ 109,\ 127,\ 151,\ 163,\ 181,\ \ldots,$$

resulting in the following triples which satisfy the given condition:

$$\begin{array}{llll} (5,7,23), & (5,13,47), & (5,19,71), & (5,43,167), \\ (5,61,239), & (5,67,263), & (5,79,311), & (5,97,383), \\ (5,109,431), & (5,127,503), & (5,151,599), & \ldots \end{array}$$

Other primes. The cases $p = 7, 11, 13, \ldots$ may be analysed in the same way; we leave the details to the reader. (The reasoning used is nearly the same in all these cases.) Many more solutions may be formed, all obeying similar relationships.

The case k = 4. Here the problem is to find primes p, q, r, s (with $p < q < r < s$) such that $pqrs$ is an integer multiple of $p + q + r + s$. We see right away that $p = 2$; for, if all four primes are odd, then their sum is even but their product is odd, so it is not possible for the product to be a multiple of the sum. Therefore the task reduces to the following: find odd primes q, r, s (with $q < r < s$) such that

$$2qrs = n(2 + q + r + s), \qquad \text{for some positive integer } n. \tag{6}$$

Arguing as earlier, we see that $2+q+r+s$ must be one of the numbers

$$2,\ q,\ r,\ s,\ 2q,\ 2r,\ 2s,\ qr,\ qs,\ rs,\ 2qr,\ 2qs,\ 2rs,\ qrs,\ 2qrs.$$

Many of these possibilities can be eliminated immediately, but some of them do yield results. We will not analyse this case in detail, but only consider one possibility, namely that $2+q+r+s=qr$. This equality leads to

$$qr-q-r+1=s+3, \quad \text{i.e.,} \quad s=(q-1)(r-1)-3. \tag{7}$$

So we must look for primes q, r such that $(q-1)(r-1)-3$ is a prime number. Here are some possibilities:

$$(q,r)=(3,11),\ (3,17),\ (3,23),\ (3,29),\ \ldots,\ (5,11),\ (5,17),\ (5,29),\ \ldots.$$

We see that there are many solutions. The above possibilities lead to the following quadruples which are all solutions of the original problem:

$$\begin{array}{llll}
(2,3,11,17), & (2,3,17,29), & (2,3,23,41), & (2,3,29,53),\\
(2,3,47,89), & (2,5,11,37), & (2,5,17,61), & (2,5,29,109),\\
(2,5,41,157), & (2,5,47,181), & (2,11,17,157), & (2,11,29,277),\\
(2,11,41,397), & (2,11,47,457), & (2,17,23,349), & (2,17,47,733),\\
(2,23,29,613), & (2,23,41,877), & (2,23,47,1009), & \ldots
\end{array}$$

There are clearly lots of solutions, and further exploration is surely possible. The problem seems rich, revealing many interesting relationships among prime numbers.

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