On the Tens Digit of a Prime Power

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In this note, we prove that the tens digit of any power of an infinite number of prime numbers is even. This is a generalization of a problem that appeared in the Regional Mathematics Olympiad in 1993.

The study of the digits appearing in the decimal expansion of a real number plays an important role in the study of various number theoretic problems. In particular, the parity of the digits in the powers of a prime number is an interesting object to study. In RMO 1993, the following problem was posed.

Problem 1 (RMO 1993)**.** Prove that the tens digit of any power of 3 is even.

In this article, we take a close look at the RMO problem and prove that the conclusion holds for an infinite number of prime numbers p. More precisely, we prove the following theorem.

Theorem 1. Let p be a prime number such that $p \equiv 3$ or $7 \pmod{20}$. Then for any integer $r \geq 1$, the tens digit of p^r is an even number.

Remark 1. Note that 3 is a prime number congruent to 3 modulo 20. Therefore, Theorem 1 is indeed a generalization of Problem 1.

Remark 2. Dirichlet's theorem for primes in arithmetic progressions asserts that if a and m are integers such that $\gcd(a, m) = 1$, then there exist infinitely many prime numbers q such that $q \equiv a \pmod{m}$. Since $\gcd(3, 20) = 1 = \gcd(7, 20)$, the theorem tells us that there exist infinitely many prime numbers of the forms $3 \pmod{20}$ and $7 \pmod{20}$. Theorem 1 now assures us that the tens digit of any power of any such prime number is even.

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Proof of Theorem 1

We give a detailed proof for $p \equiv 3 \pmod{20}$. For the residue class $7 \pmod{20}$, the proof follows almost the same line of argument.

Let $r \geq 1$ be an integer and let $p \equiv 3 \pmod{20}$ be a prime number. Then $p = 20m + 3$ for some integer m. Since we are dealing with the tens digit of p^r, we shall be concerned with $p^r \pmod{100}$. Therefore, it is convenient to put $k = 2m$ and use the fact that k is an even integer.

The proof is by induction on r. For $r = 1$, the tens digit of p^r is even because $p = 10k + 3$ with k even. Therefore, the theorem holds true for $r = 1$. Now, suppose that the tens digit of p^r is even for some integer $r \geq 1$.

Let $p^r = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_s \cdot 10^s$ be the decimal expansion of p^r (where s is some positive integer). Then we have

$$\begin{aligned} p^{r+1} &= (10k+3)(a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \ldots + a_s \cdot 10^s) \\ &\equiv 3a_0 + 30a_1 + 10k \cdot a_0 \pmod{100}. \end{aligned} \tag{1}$$

By the induction hypothesis, a_1 is even. Also, we note that since $p \equiv 3 \pmod{10}$, we have $p^{r+1} \equiv a_0 \equiv 1, 3, 7, 9 \pmod{10}$. We consider the four cases separately.

Case 1, $\mathbf{a_0 = 1}$. Then $p^{r+1} \equiv 3 + 30a_1 + 10k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$3 + 30a_1 + 10k = \begin{cases} 10k + 3 & \text{if } a_1 = 0, \\ 10k + 63 & \text{if } a_1 = 2, \\ 10k + 123 & \text{if } a_1 = 4, \\ 10k + 183 & \text{if } a_1 = 6, \\ 10k + 243 & \text{if } a_1 = 8. \end{cases}$$

Since k is even and the tens digits of $3, 63, 123, 183$ and 243 are all even, we conclude that the tens digit of p^{r+1} is even.

Case 2, $\mathbf{a_0 = 3}$. Then $p^{r+1} \equiv 9 + 30a_1 + 30k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$9 + 30a_1 + 30k = \begin{cases} 30k + 9 & \text{if } a_1 = 0, \\ 30k + 69 & \text{if } a_1 = 2, \\ 30k + 129 & \text{if } a_1 = 4, \\ 30k + 189 & \text{if } a_1 = 6, \\ 30k + 249 & \text{if } a_1 = 8. \end{cases}$$

Again we note that the tens digits of $9, 69, 129, 189$ and 249 are all even. Hence the tens digit of p^{r+1} is even.

Case 3, $\mathbf{a_0 = 7}$. Then $p^{r+1} \equiv 21 + 30a_1 + 70k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$21 + 30a_1 + 70k = \begin{cases} 70k + 21 & \text{if } a_1 = 0, \\ 70k + 81 & \text{if } a_1 = 2, \\ 70k + 141 & \text{if } a_1 = 4, \\ 70k + 201 & \text{if } a_1 = 6, \\ 70k + 261 & \text{if } a_1 = 8. \end{cases}$$

Since the tens digits of $21, 81, 141, 201$ and 261 are all even, we conclude that the tens digit of p^{r+1} is also even.

Case 4, $\mathbf{a}_0 = 9$. Then $p^{r+1} \equiv 27 + 30a_1 + 90k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$27 + 30a_1 + 90k = \begin{cases} 90k + 27 & \text{if } a_1 = 0, \\ 90k + 87 & \text{if } a_1 = 2, \\ 90k + 147 & \text{if } a_1 = 4, \\ 90k + 207 & \text{if } a_1 = 6, \\ 90k + 267 & \text{if } a_1 = 8. \end{cases}$$

Since the tens digits of $27, 87, 147, 207$ and 267 are all even, we conclude that the tens digit of p^{r+1} is also even.

Therefore, by the method of mathematical induction, we conclude that the tens digit of any power of p is even. This completes the proof of Theorem 1. $\square$

The interested reader can enquire for which prime powers the tens digit is divisible by 4, by 8, and so on.

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