Secondary

Problem Corner

Solutions to Two Problems

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We present solutions to two of the problems which appeared in the Senior Problem Set of the November 2020 issue of *At Right Angles*. Both have been submitted by Praneetha Kalbavi of Class XI, The Learning Center PU College, Mangalore.

Problem IX-3-1

Consider the quadratic function $f(x) = x^2 + bx + c$ defined on the set of real numbers. Given that the zeros of f are two distinct prime numbers p and q, and $f(p-q) = 6pq$, determine the primes p and q, and the function f.

Solution. As p, q are the roots of $x^2 + bx + c = 0$, we have $p^2 + bp + c = 0$ and $q^2 + bq + c = 0$. Hence by subtraction,

$$p^2 - q^2 + b(p-q) = 0, \quad \therefore\ p^2 + b(p-q) = q^2.$$

Combining this with the given fact that $6pq = f(p-q) = p^2 + q^2 - 2pq + b(p-q) + c$, we get

$$8pq = 2q^2 + c,$$

giving $c = 2q(4p-q)$. Now, using the familiar equalities for sum and product of the roots of a quadratic equation, we get:

$$c = pq, \quad \therefore\ 7pq = 2q^2, \quad \therefore\ 7p = 2q.$$

Since p and q are prime numbers, the last equality can only be satisfied if $p = 2$ and $q = 7$. This gives $c = 14$. Hence $f(x) = x^2 - 9x + 14$. □

Keywords: Quadratic function, prime number, equation, solution

Problem IX-3-5

Solve for real x:

$$4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} = 1.$$

Solution. To start with, note that we must have $\frac{1}{2} - 2x^2 \geq 0$, and therefore $x^2 \leq \frac{1}{4}$, i.e., $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Now if $x \geq 0$, then $a^x \geq 1$ for any $a > 1$, hence $4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} > 1$. So the given equation has no solution with $0 \leq x$.

Next, note that $4^{-1/2} + 9^{-1/2} + 36^{-1/2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, and that $\sqrt{\frac{1}{2} - 2x^2} = 0$ when $x = -\frac{1}{2}$. This means that $x = -\frac{1}{2}$ solves the given equation. So $x = -\frac{1}{2}$ is a solution of the equation.

Finally, suppose that $-\frac{1}{2} < x < 0$. Then $4^x + 9^x + 36^x > \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$, i.e., $4^x + 9^x + 36^x > 1$, and therefore $4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} > 1$. So the given equation has no solution with $-\frac{1}{2} < x < 0$.

It follows that the only solution to the given equation is $x = -\frac{1}{2}$. □

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