

Exploring the Nine-point Circle: Conjecture making and proof using Dynamic Geometry Software

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Digital technologies provide many opportunities for learning mathematics through exploration and visualization. Dynamic Geometry Software (DGS) are a special class of digital tools that have made a significant impact in mathematics education across the world and have led to many research studies. In the article [1] in the July 2020 issue of *At Right Angles*, we demonstrated how an open-source DGS like GeoGebra functions as a conjecture making tool and can be used by students to explore mathematical concepts. In particular, the article described how grade VI students used GeoGebra to conjecture the angle sum property of a triangle.

In this article we shall describe how grade IX students visualised the Euler's circle (more popularly known as the nine-point circle) of a triangle, using GeoGebra. The journey undertaken by them as they constructed the figure, made conjectures and finally arrived at the proof will be discussed.

Background of the Study

The 34 students, who participated in the study had no prior experience in using GeoGebra or any other dynamic geometry software. They came from two schools in Mumbai and had volunteered for the study. In their regular mathematics classes in school, they had studied the properties of triangles, quadrilaterals and circles. They were also familiar with the proofs of certain theorems such as the midpoint theorem of a triangle and others related to circles and quadrilaterals. The exploration was conducted over two 2-hour sessions in a computer lab. The students were divided into pairs and each pair of students was assigned to a computer, which had GeoGebra. Before the

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sessions, students were informed (by their teacher) that they would explore a geometrical problem using a digital tool. The first 30 minutes (of the first session) was used by the teacher to familiarize the students with the basic construction tools of the GeoGebra toolbar, namely the **Move tool**, **Point Tool**, **Line tool**, **Polygon tool** and the **Circle tool**. It was interesting to observe how students enthusiastically explored the various features within each tool, on their own. Indeed, they asked for extra time to “play” with the tools.

After the students had gained some initial familiarity with the construction tools, the teacher posed the following tasks. The tasks were given to them in a worksheet in which they were required to record their observations and explain their reasoning.

Task 1: Draw a triangle ABC on your GeoGebra screen and do the following.

- (i) Mark the midpoints D, E and F of the sides BC, CA and AB respectively.
- (ii) Construct the three altitudes of the triangle and label the orthocenter as H.
- (iii) Mark the feet of these altitudes. Let the foot of the altitude from A to BC be I, from B to CA be J and from C to AB be K.
- (iv) Draw the segments connecting the orthocenter H to the vertices of the triangle A, B, C and mark their midpoints. Label the midpoints of HA, HB and HC as L, M and N respectively.

Task 2: Drag any vertex of the triangle ABC (you may make it an acute, obtuse or a right-angled triangle). What do you observe about the points D, E, F, I, J, K, L, M and N? Can you make a conjecture regarding these nine points? Can you prove your conjecture?

The GeoGebra Construction

The following steps will enable the reader to go through the construction process in GeoGebra (as given in Task 1).

Step 1: Use the **Polygon tool** to draw a triangle ABC in the **Graphics view**. Make sure that the **Algebra view** is on. The latter will highlight all elements of the geometrical objects constructed in the **Graphics view**.

Step 2: Using the **Midpoint or Centre tool** (available within the **point tool**), click on BC, CA and AB. The midpoints will automatically be marked D, E and F respectively. See Figure 1.

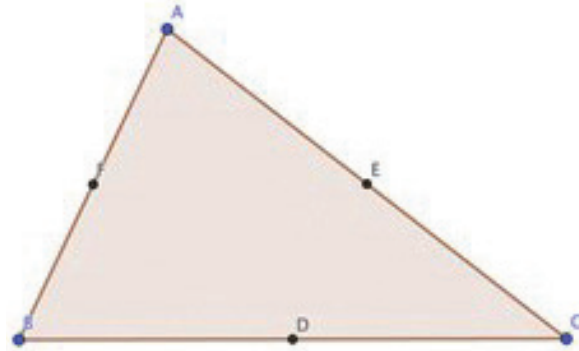


Figure 1. Triangle ABC and the midpoints of the sides.

Step 3: Now to construct the altitudes, select the **perpendicular line tool**. Click on A first, and then anywhere on the line BC. This will produce the altitude from A to BC. Similarly, draw the other two altitudes of the triangle.

Step 4: Select the **Intersect tool** (available within the **Point tool**) and mark the feet of the altitudes. (Clicking on the point of intersection produces a point, which can be renamed with a right-click.) Also, use the **Intersect tool** to mark the point of concurrency of the three altitudes as H. This is the orthocenter of triangle ABC. See Figure 2.

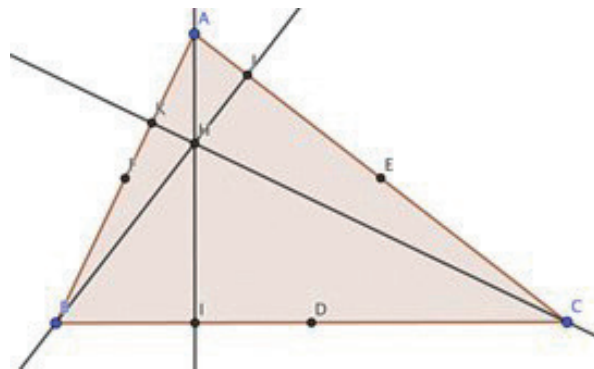


Figure 2: The Perpendicular line option is used to construct the three altitudes.

Step 5: All the elements of the constructed figure (points, lines and polygon) will appear in the **Algebra view**. Click on the blue dots alongside the lines (altitudes) to hide them.

Step 6: Using the **Segment tool** draw the segments HA, HB and HC. Mark their midpoints as L, M and N respectively using the **midpoint tool**. See Figure 3.

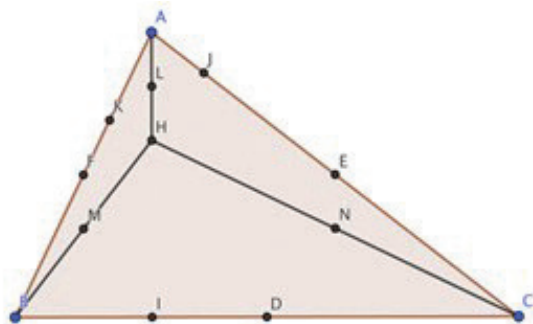


Figure 3. The segments HA, HB and HC and their midpoints L, M and N.

Step 7: Drag vertex A (or any other vertex of the triangle) and observe what happens to the nine points D, E, F, I, J, K, L, M and N.

Students' Explorations

After completing task 1, the students dragged the vertices of the triangle while at the same time observing the nine points as the triangle changed from being acute-angled to right-angled and then obtuse-angled. Dragging led many students to make the conjecture that the nine points lie on a circle. To many, this was more obvious when the triangle ABC was acute-angled. One student remarked, "The points appear to be swimming around in a circle. It looks like a circle at least when the triangle has acute angles." Some students were eager to verify their conjecture and enquired "but can we check if these points do lie on a circle?"

The teacher encouraged them to search the menu of the circle tool, which has a 'circle through 3 points' option. This produces a circle passing through any three selected points. The option led to much excitement as the students quickly used this tool to select any three points (among D, E, F, I, J, K, L, M and N) to obtain the

circle passing through them. Figure 4 shows the nine-point circle when triangle ABC is acute-angled. After further exploration another student observed, "Whichever three points I choose for a given triangle ABC, I always get the same circle. So this must be the nine-point circle we are looking for!" It was a high point for the teacher who realized that by now students were already engaged in reasoning and argumentation.

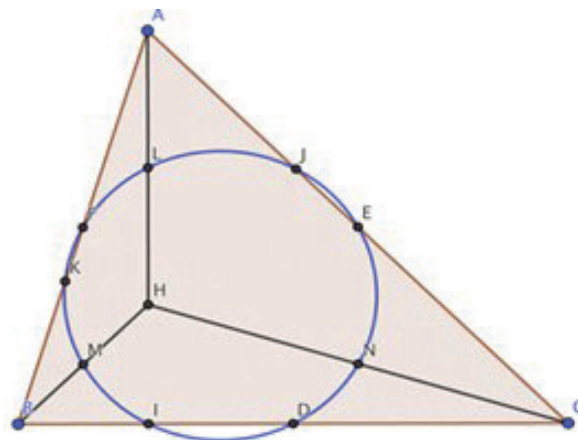


Figure 4. The nine-point circle of the triangle ABC, drawn using the Circle tool.

By dragging vertex A, the triangle ABC can be made right-angled at B as shown in Figure 5. In this case, the feet of the altitudes, I and K, the point M and the orthocenter H, all coincide with the vertex B. During the exploration, some students commented on how "some points have disappeared and only five points are visible." Others reinforced this with the argument "some points come on top of others (meaning that they

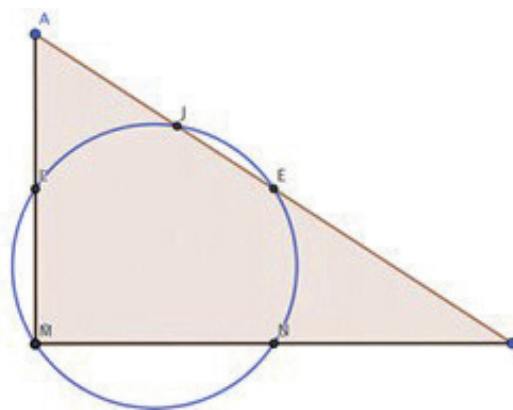


Figure 5. The nine-point circle when ABC is a right-angled triangle.

coincide).” Figure 5 shows the circle passing through the five points M, L, J, E and N. This may be considered as a degenerate case of the nine-point circle. It led to a discussion among the students as to which points coincide and why. For example, with some scaffolding by the teacher, the students concluded that the legs of a right-angled triangle are also its altitudes and hence the orthocenter (H) coincides with the vertex B.

When the triangle becomes obtuse-angled, the segments HA, HB and HC and the orthocenter H lie outside the triangle, but the nine-point circle remains. (All nine points are now visible as shown in Figure 6.)

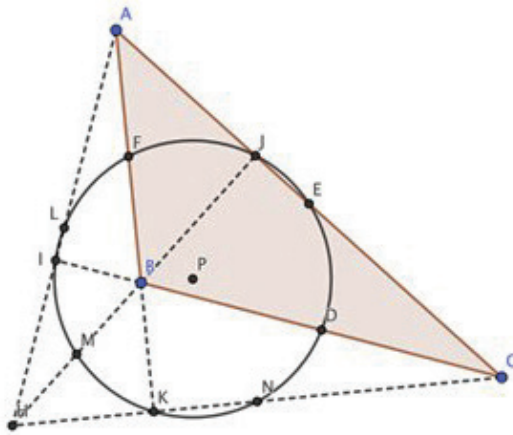


Figure 6. The nine-point circle when ABC is an obtuse-angled triangle.

Dragging as a tool leading to the conjecture

In GeoGebra and other DGS, dragging as a tool lends many opportunities to make and verify conjectures. In a dragging episode, the user observes invariant properties of the object and simultaneously experiences the varying aspects. Researchers have corroborated this unique feature of a DGS.

According to Leung [3]

..... one of DGS power is to equip us with the ability to retain (keep fixed) a background geometrical configuration while we can selectively bring to the fore (via dragging) those parts of the whole configuration that interested us in a mathematical thinking episode.

In the nine-point circle exploration, dragging a vertex of the triangle led to variations in the side lengths, angle measures and positions of the nine points. However, the existence of the circle passing through the nine points remained invariant. This contrasting experience (of the varying and the invariant) was instrumental in enabling the students to make the conjecture. While the students delighted in seeing the circle on their GeoGebra screen, the teacher asked: “how should we prove this conjecture?” One student pointed out that the nine-point circle is “always present for every kind of triangle” which is enough to substantiate the claim, while another felt “this is not a proof.” Finally, the students agreed that a formal proof of the fact that all the nine points were on a circle, was required.

Towards proof

Since the students were in grade IX, they were familiar with properties of rectangles, parallelograms and circles. They had also studied the midpoint theorem in their regular classes. Fortunately, this knowledge was enough to enable them to explore the proof of the existence of the nine-point circle. In this part of the session, the teacher had to play a more active role of facilitation and scaffolding.

The first step was to prove that the quadrilateral FENM (marked in green in Figure 7) forms a rectangle. Note that in triangle ABC, F and E are the midpoints of AB and AC respectively. Hence by midpoint theorem, FE is parallel to BC and equal to half of it. Similarly, in triangle HBC, M and N are the midpoints of HB and HC. Thus MN is parallel to BC and equal to half of it. With this line of argument, the students concluded that FE is equal and parallel to MN and hence FENM is a parallelogram. “But is it a special kind of a parallelogram?” the teacher asked. When one student remarked “it looks like a rectangle” another responded, “but we need to prove it.” Once again it was a high point for the teacher. The students were actively engaged in the process of proving their arguments. At

the same time, they were also recording their observations and reasoning on their worksheets.

In triangle ABH , FM is parallel to AH (by midpoint theorem). Also since AH is perpendicular to BC (it is an altitude of triangle ABC), it implies that FM is perpendicular to MN . By similar arguments, students were able to show that EN is perpendicular to MN . Thus $FENM$ is a rectangle. Since a rectangle is a cyclic quadrilateral, it was concluded that the points F, E, M and N lie on a circle with diameter FN (this is so because angle FEN is a right angle, as is angle FMN). See Figure 7.

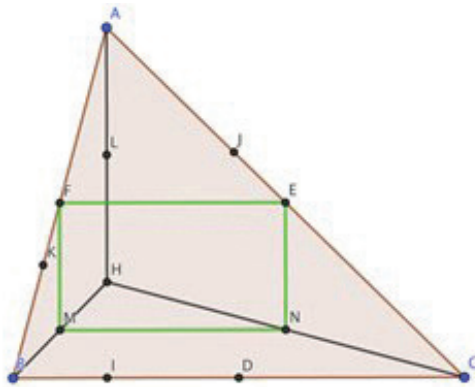


Figure 7. The quadrilateral $FENM$ is a rectangle with diagonal FN .

Using similar arguments students proved that $FLND$ (marked in blue in Figure 8) is also a rectangle. The circle passing through the points F, L, N, D also has FN as a diameter. As the two circles share a diameter, they are identical. Thus the points D, E, F, L, N and M lie on the same circle. Students also observed that FN , a diameter of the circle, is also the common diagonal of the rectangles $FENM$ and $FLND$. FN is marked as a

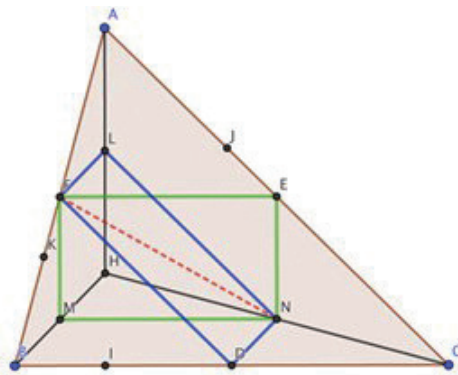


Figure 8. The quadrilateral $FLND$ is a rectangle with diagonal FN .

red dotted line in Figure 9. Also, the midpoint of FN is the centre of the circle as shown in Figure 9.

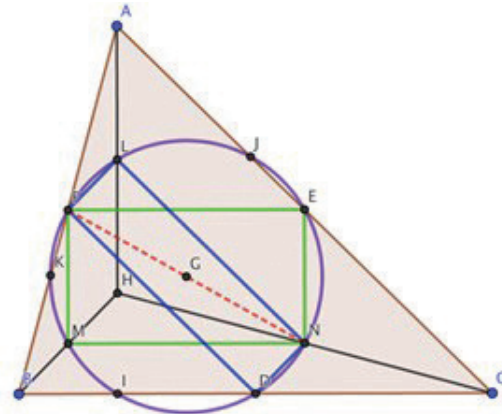


Figure 9. The nine-point circle has the diagonal FN or LD as a diameter.

Finally, it was required to prove that the feet of the altitudes, namely, I, J and K also lie on the circle (with diameter FN), which contains the six points. GeoGebra offers the option to ‘hide’ or ‘unhide’ parts of a figure. This useful feature is a great advantage over paper-pencil constructions. At this stage of the exploration, the rectangles and the line segments HA, HB and HC were ‘hidden.’ The diagonal LD of the rectangle $FLND$ (which is also a diameter of the circle) was drawn. It was easy to see (as shown in Figure 10) that the angle LID is a right angle. This implies that I lies on the circle with diameter LD (the converse of the theorem which states that angle in a semicircle is a right angle) and is therefore also on the circle. Similarly, it was shown that J and K also lie on the circle. This proved the conjecture about the nine-point circle.

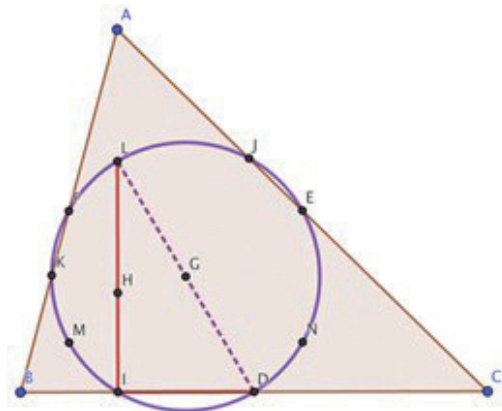


Figure 10. Angle LID is a right angle implying that point I lies on the circle with LD as diameter.

DGS as an amplifier and organiser

One of the important roles of the use of digital technology in mathematics learning is its use as an *amplifier* and *organiser* of mental activity [5]. When technology takes over time-consuming computations and operations (that might be done by hand), it assumes the role of an amplifier. For example, a digital tool can quickly and accurately produce graphs or generate a table of values leading the student (and teacher) to focus on observing patterns and developing insight. As an amplifier, technology supports students' mathematical thinking. However, when technology is used as a reorganizer, it enhances students' mathematical thinking by giving them access to higher-level processes that would be difficult to achieve in a paper-pencil mode.

In the exploration of the nine-point circle by grade IX students, GeoGebra played a pivotal role as an amplifier and reorganizer. When asked, how they viewed the GeoGebra exploration in comparison to a compass and ruler construction, many students felt that constructions in GeoGebra were 'faster', 'more accurate', 'dynamic' and 'easy to understand'. GeoGebra quickly produced constructions allowing students to focus on the properties and measurements of parts of the figure (such as lengths of segments), from the Algebra view. Here GeoGebra played the role of an amplifier. However, once the construction was completed, the students began to focus on changes in the figure brought about by dragging. Dragging enabled them to experience varying appearances of the triangle ABC and look for invariant properties, leading to the conjecture about the nine-point circle. Here, GeoGebra played the role of an organiser by extending student's mathematical thinking. For example, by exploring the acute angle case (figure 4), a majority of the students were able to conjecture that the nine points lie on a circle. Students asked many 'what if?' and 'why?' questions and tried to justify their arguments with reasoning. GeoGebra also motivated the need to find a proof of the invariant nine-point circle. According to a study by King and Schattschneider [2], a dynamic

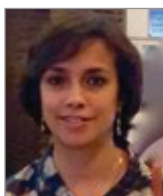
geometry software cannot produce proofs, but it can motivate the desire for proof in students. Appropriate facilitation by the teacher led students to write the proof using their knowledge of properties of quadrilaterals and the midpoint theorem.

Conclusion

The outcome of the study was illuminating as the grade IX students, with no prior knowledge of GeoGebra, were able to use it to explore a geometrical problem, make conjectures and finally arrive at a proof. Throughout the session, the students worked in pairs. Within each pair, they took turns at working with GeoGebra on their computer. While one student worked on the computer, the other recorded their common observations on their worksheet. They soon realized that GeoGebra would help them make observations but when it came to writing the proof, they needed to provide sound justifications for their arguments. GeoGebra empowered the students by extending their mathematical thinking and by enabling them to perform geometrical explorations beyond the scope of grade IX. The role of the teacher (who was also the researcher) was critical, as she had to steer the students' explorations without giving away answers, making sure that there was enough opportunity for collaboration and discussion. The teacher encouraged discussions among students throughout the two sessions. Overall GeoGebra led to a satisfying combination of technology use and engagement with argumentation and proof. Further, the positive feedback from students was highly encouraging as they expressed the desire to undergo similar sessions in other topics. Lingefjord and Ghosh [4] report their findings of a similar study where students of grade IX explored the Euler line (collinearity of the centroid, circumcentre and orthocenter of a triangle) using GeoGebra. The encouraging results of such studies point to the possibilities and affordances in integrating Dynamic Geometry Software for teaching and learning mathematics.

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