Secondary

Problem Corner

A Geometry Problem from the Putnam 2019 Competition

RAKSHITHA &
P N SUBRAHMANYA

In this article, we study a geometry problem adapted from the Putnam exam of 2019. (The William Lowell Putnam Mathematical Competition or the 'Putnam Competition' is an annual mathematics competition for undergraduate college students enrolled at institutions of higher learning in the United States and Canada.)

Problem. In triangle ABC, let G be the centroid and I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB, and $\tan \beta/2 = 1/3$. Find α.

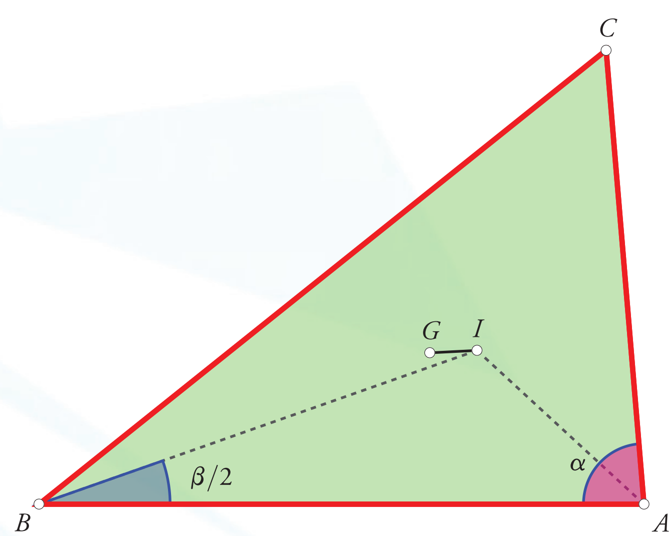

Figure 1.

Keywords: Putnam, triangle centres, angles, similar triangles, trigonometry, ratios

Solution. We use approaches from coordinate geometry, trigonometry and pure geometry and argue as follows.

- We are told (see Figure 1) that $\tan\beta/2 = 1/3$. From this it follows that
$$\tan\beta = \frac{2\tan\beta/2}{1-\tan^2\beta/2} = \frac{2/3}{1-1/9} = \frac{3}{4}.$$
- We start by assigning coordinates as follows: $B=(0,0)$, $A=(a,0)$. Note that AB lies on the x-axis.
- As $\tan\measuredangle CBA = 3/4$, we may fix the scale of the coordinate axes so that $C=(4,3)$.
- Using this, we find that the y-coordinate of the centroid G is 1.
- Since $IG \parallel AB$, it follows that the y-coordinate of I too is 1.
- Since $\tan\measuredangle IBA = 1/3$, it follows that the x-coordinate of I is 3; so $I=(3,1)$.
- Since the distance from I to AB is 1, it follows that the radius of the incircle is 1.

We may now follow two possible approaches.

First approach: We use the formula connecting radius of the incircle and area of the triangle:
$$\text{Radius of incircle} = \frac{\text{Area of triangle}}{\text{Semi-perimeter of triangle}}.$$

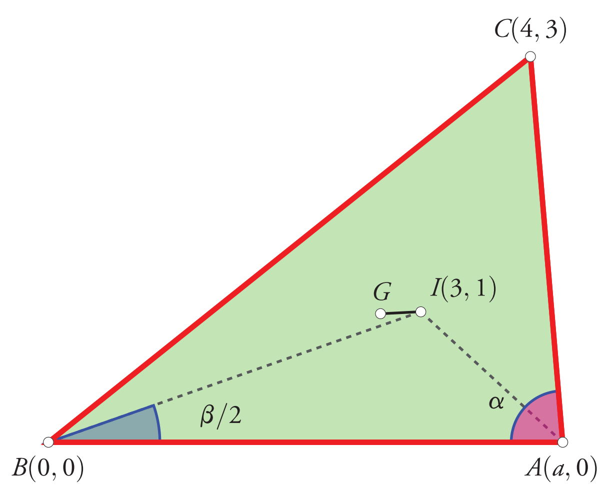

Figure 2.

Here we have (see Figure 2):
$$\text{Area} = \frac{3a}{2},$$
$$\text{Perimeter} = a+5+\sqrt{(a-4)^2+3^2},$$
$$\text{Radius} = 1.$$

Hence:

$$\begin{aligned} a+5+\sqrt{(a-4)^2+3^2} &= 3a, \\ \therefore\ (2a-5)^2 &= (a-4)^2+3^2, \\ \therefore\ 3a^2 &= 12a, \end{aligned}$$

giving $a = 4$. (The solution $a = 0$ is not meaningful.) Hence $A = (4, 0)$. Since $C = (4, 3)$, it follows that $CA \perp AB$. Thus $\alpha = 90°$. $\square$

Second approach: Here we think geometrically rather than algebraically.

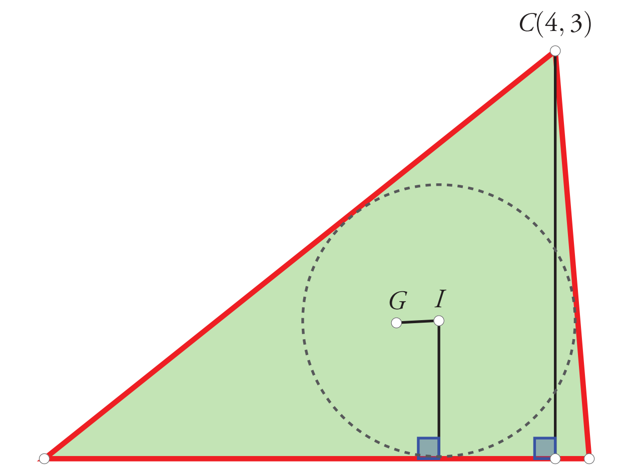

Figure 3.

Draw a perpendicular CD from C to AB. Also draw the incircle (centre I, radius 1) of triangle ABC. Since the x-coordinate of I is 3, and line CD has equation $x = 4$, it follows that the incircle touches CD. But it also touches line CA, by definition of an incircle. This means that both CD and CA are tangent to the incircle. Therefore they coincide, which means that D coincides with A. This implies that $\measuredangle CAB$ is a right-angle, i.e., $\alpha = 90°$. $\square$

References

1. 2019 William Lowell Putnam Mathematical Competition Problems, https://www.maa.org/sites/default/files/pdf/Putnam/2019/2019PutnamProblems.pdf

RAKSHITHA is a math enthusiast studying in class 12 at The Learning Centre, PU College, Mangalore. She has a deep interest in Analysis, Number Theory and Combinatorics. Other than Mathematics, she is fascinated by the sciences and wishes to pursue pure research in the future. She participated in the Indian National Mathematics Olympiad in 2019 and 2020. She may be contacted at krraja1974@rediffmail.com.

P N SUBRAHMANYA has been teaching mathematics at the high school and pre-college level at The Center for Advanced Learning, Mangalore for the past 10 years. He is keenly interested in promoting non-routine problem solving among his students, particularly students of rural areas. His other interests include history and Kannada literature, especially poetry. He may be contacted at subrapadekal17@gmail.com.