Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Mathematics of Securing a Box With a Single Rubber Band*

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In this article, we study the configuration of a rubber band wrapped around a (sweet) box. The rubber band is a loop trying to attain minimum length. We, therefore, study such curves, also known as closed geodesics, on a cuboid which is the typical shape of such a box. The arguments should be accessible even to a high school student, while the conclusions have elements of surprise and hence interest.

Introduction

One must have noticed rubber bands around boxes wrapped in different configurations, as shown in *Figure* 1. The one in *Figure* 1(c) particularly catches attention since a single band visits all the faces of the box without intersecting itself. It is interesting to ask—"What is its configuration?". Here we answer this question by noting that the minimum energy will correspond to the minimum length of the rubber band.

Geometrically the box is a cuboid. Let l, b, and h respectively denote the length, breadth, and height of the box. So we are now

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interested in geodesics (shortest paths) on the surface of a cuboid.

Keywords
Rubber band, cuboid, closed geodesics, straight line, geometric mean

This problem is similar to that of finding the shortest path for an insect crawling from one point on one wall (/floor/roof) of a room to another point on another wall. This is known as the "spider and fly problem" [1]. But, we are interested in *closed* geodesics, where the path that goes round the box, apart from being shortest among its neighbors, also starts and ends at the same point.

Meeting this requirement in itself is easy. One can wrap the rubber band around the box by covering the shorter two dimensions as in *Figure* 1(a). If l is the largest dimension, then the length of the rubber band in this configuration would be $2(b+h)$. While this is indeed the shortest closed path around the box, by experience, we know that this configuration may not keep the packing tight because the rubber band does not visit all the faces.

A quick fix would be to run the rubber band as shown in *Figure* 1(b). This figure shows a *single* band, two parts of which are looped around each other on the top of the box. However, the one which shopkeepers commonly use, as in *Figure* 1(c) is even better, in the sense that the length of the rubber band would be smaller, as we will see below. We come back to the configuration in *Figure* 1(b) at the end and now venture to obtain the closed geodesics in *Figure* 1(c).

Examining *Figure* 1(c), two out of the three dimensions get traversed *once*, back and forth, in a complete journey along the rubber band. The third dimension gets traversed *twice*, back and forth, in that journey. We call this third dimension h (the shortest edge in *Figure* 1(c)) and the other two as l and b. This choice also means that the rubber band never crosses the 4 edges of dimension h but crosses all the other 8 edges.

We later encounter cases with two choices for h. But, it will be shown that the longest edge can never take the role of h. Hence, without loss of generality, let us choose the largest dimension as l for the rest of our discussion. This implies that the rubber band certainly crosses the four edges of length l. If you start from the point where it crosses one such edge of length l, the next edge can

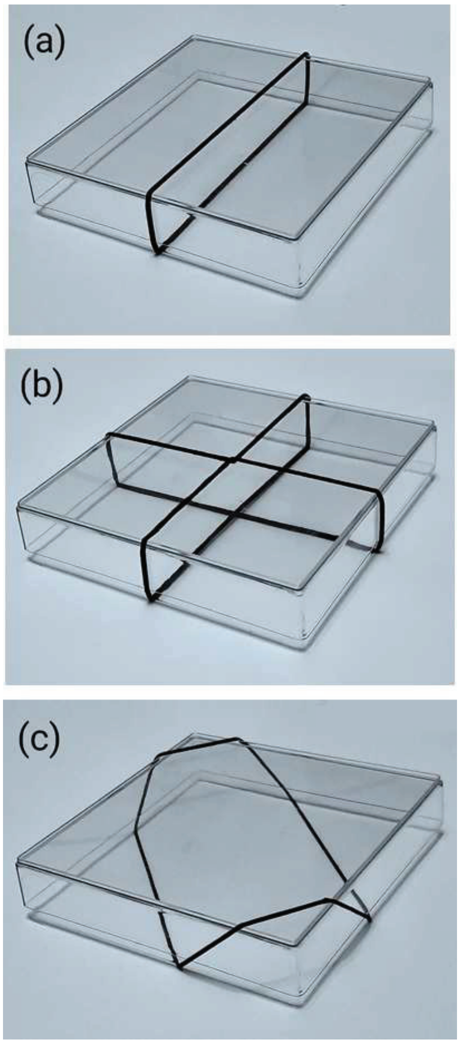

Figure 1. Different types of wrapping of the rubber band around a box. Here we focus on the configuration in (c).

either be of length l or of length b.

It is assumed here throughout that there is no friction anywhere. The role of friction in a real situation is briefly discussed at the end. Now, we obtain the length of the rubber band along with the number of distinct ways to wrap it for a given l, b and h.

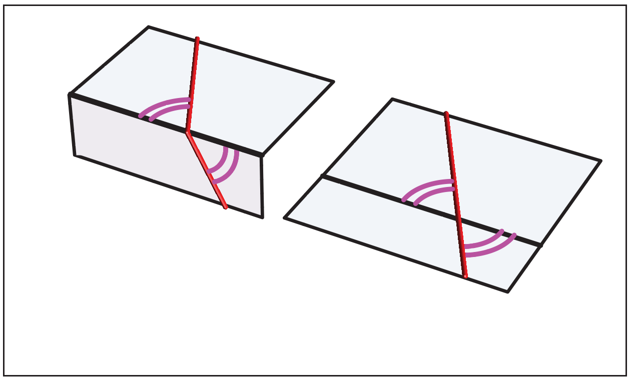

Figure 2. Illustration of requirement 2 by opening up a set of adjacent faces around their common edge onto a plane.

Requirements for a Geodesic

We begin by listing the basic requirements for a geodesic, whether it is open or closed, on a cuboid. The first one, which is obvious from the minimization of length, is regarding the nature of geodesic within a given face.

Requirement 1. *The rubber band, on any given face follows a straight line.*

The next one tells us how the rubber band bends over an edge:

Requirement 2. *The rubber band, makes the same angle with an edge, both before and after crossing it.*

If a pair of faces having a common edge are unfolded around it to form a single planar surface, while the rubber band crossing that edge remains glued to them, then the rubber band should follow a single straight line after unfolding to minimize its length.

The justification for this is as follows: If a pair of faces having a common edge are unfolded around it to form a single planar surface, while the rubber band crossing that edge remains glued to them, then the rubber band should follow a single straight line after unfolding to minimize its length. The two angles referred in requirement 2 will then form vertically opposite angles after unfolding and hence are equal. This is illustrated in *Figure* 2.

Shape and Length of the Rubber Band

Based on these requirements, the strategy to find the configuration of the rubber band in *Figure* 1(c), would then be to "open out the

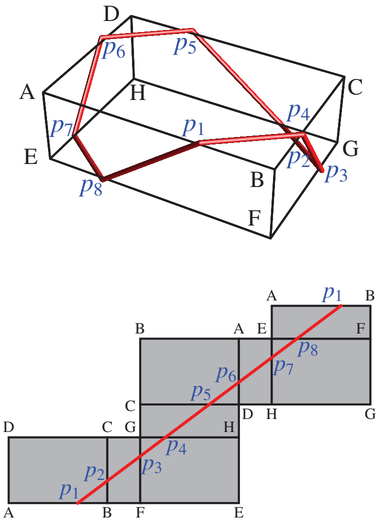

Figure 3. Rubber band around the box in its usual (folded) form and the same after unfolding as explained in the text. Note that the rubber band visits a pair of opposite faces (ABCD and EFGH) twice. So there are effectively 8 faces in action.

box" onto a flat surface, and draw a straight line! This is shown in *Figure* 3.

As per the conventions stated earlier, the dimension AB in *Figure* 3 (being the longest) is l and the dimension BF (having been traversed twice) is h. The other dimension BC will then be b. We choose a starting point on an edge, say p_1 on AB, and a point on the next edge to visit, say p_2 on BC. The path from p_1 to p_2 and the subsequent path of the rubber band gets dictated by the requirements listed in the previous section. Given that the rubber band has to return to the initial point after visiting all the faces, the choice of p_2 on BC is unique. Using the Pythagoras theorem, the total length of the rubber band, R, is

$$R = 2\sqrt{(l+h)^2 + (b+h)^2}. \tag{1}$$

This solution is evidently better than that corresponding to *Figure* 1(b), where R would have been $2\,((l+h)+(b+h))$. After all, the length of the hypotenuse (the rubber band itself in the figure) in a right angled triangle is always less than the sum of the other two sides!

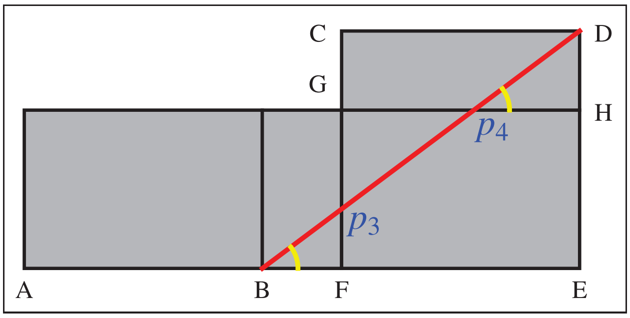

Figure 4. The situation when the rubber band in *Figure* 3 is pushed parallel to extreme right of itself.

From *Figure* 3 one can also see that the rubber band can be transported parallel to itself without changing its length R. This means p_1 moves along AB and other points also move accordingly. This freedom implies that its configuration corresponds to a neutral equilibrium. Thus, we encounter a family of solutions that are degenerate. In a real situation, however, the friction between the box and the rubber band stops it from moving.

Also, observing *Figure* 3, a symmetry uncovers itself: *The full journey of the rubber band around the box can be split into two identical ones.* This symmetry continues to hold even if the band is transported parallel to itself. This also means that the situation at point p_1 is same as that at p_5. A similar statement holds for the pairs (p_2, p_6), (p_3, p_7) and (p_4, p_8).

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Distinct Ways of Wrapping the Rubber Band

We saw that the rubber band can be moved parallel to itself. However, if one keeps moving like this, then beyond a certain point it slips off one of the edges. For example, in *Figure* 3 the rubber band can be moved to its right only until the points marked p_1 and p_2 fuse at B (and so do p_5 and p_6 at D). This extreme case is shown in *Figure* 4, where the symmetry noted just now has allowed us to keep only half of the rubber band's complete journey. The angles shown are corresponding angles and hence are equal. This means $\mathrm{DH}/p_4\mathrm{H} = p_3\mathrm{F}/\mathrm{BF}$. Therefore,

$$\frac{h}{l - \mathrm{G}p_4} = \frac{b - \mathrm{G}p_3}{h}. \tag{2}$$

Cross multiplying, this imposes a condition on l, b and h to host a solution, namely

$$h^2 < lb. \tag{3}$$

In other words, h has to be smaller than the geometric mean of l and b.

This condition immediately implies that the longest dimension of the cuboid can never take the role of h, as mentioned earlier. Whereas, when h is the shortest dimension, the above condition certainly gets satisfied. This means that the rubber band can always be wrapped around the box in at least one way. Interestingly, in some cases, the condition (3) above can also be satisfied by choosing the dimension of intermediate size as h. In such cases, the rubber band can be wrapped around the box in *two* different ways.

The cube is an exception where condition 3 cannot be satisfied [2].

As an example, consider $l : b : h = 3 : 2 : 1$. Then there is only one kind of solution, where the shortest dimension gets the role of h. Whereas, if $l : b : h = 10 : 3 : 2$, then there are two ways to secure the rubber band around the same box by interchanging the roles of b and h, as shown in *Figure* 5. Here, either the shortest or the intermediate dimension can play the role of h. In general, we get two solutions whenever the intermediate dimension is smaller than the geometric mean of the other two.

Hence, the number of distinct ways in which the wrapping can be done to respect condition (3) above is generically at least one and at most two. This number for any l, b and h is shown in *Figure* 6. As one would expect, this figure is symmetric around

The readers are encouraged to draw 'open-box diagrams' for both of these possibilities as in *Figure* 3.

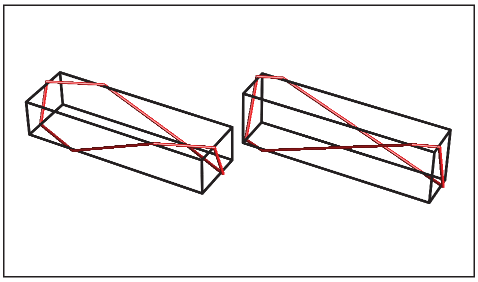

Figure 5. Two distinct ways of wrapping the rubber band around the same cuboid by interchanging the roles of b and h. The ratio $l : b : h$ for the left figure is $10 : 3 : 2$ and for the right one it is $10 : 2 : 3$.

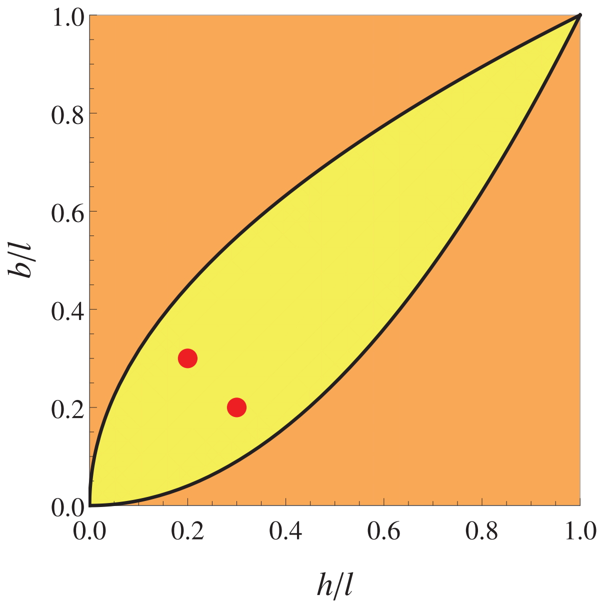

Figure 6. The number of ways in which the rubber band can be wrapped around the box (not counting the freedom to slide over an edge). If l, b, and h are chosen to fall in the orange region, then there is exactly one kind of solution. On the other hand, if they fall in the yellow region, then there are exactly two kinds of solutions. The two red dots correspond to the two solutions drawn in *Figure* 5.

the $h = b$ line. Both the axes range from zero to one as the largest dimension is taken as l. The reader can check that the equations of the two boundaries are $y = x^2$ and $x = y^2$.

When two solutions are possible, then from eqn. (1), the one with $b > h$ always gives a smaller length R for the rubber band than the one with $b < h$.

In the discussion around *Figure* 3, the point p_2 was located on the side BC rather than AD. It is another interesting exercise to check that both possibilities exist if $\mathrm{A}p_1$ and $\mathrm{B}p_1$ are both greater than $h(l + h)/(b + h)$, which can happen only if $b/l > (h/l) + 2(h/l)^2$. In that case, the second solution can be obtained from the first by first sliding the rubber band parallel to itself, followed by one of the three reflections discussed earlier.

Directions to Explore Further

We have thus obtained the shortest length configuration(s) of the rubber band, visiting all the faces of a frictionless cuboid. This problem, although interesting, may appear to be an isolated elementary exercise. A smooth analogue of a cuboid is an ellipsoid, and the problem of geodesics on an ellipsoid is far from elementary and is regarded as one of the achievements of the great 19th-century mathematician Jacobi [3].

Our problem is also connected to more advanced topics from the recent literature. Reference [4] explores closed geodesics on the dodecahedron. This is the regular solid with 12 pentagonal faces—the geodesic is closed but does not visit all of them. The closed geodesics form a rich set!

While friction may appear to be a topic from elementary physics, its mathematical and experimental consequences are far from trivial and are explored in [5]. In our problem, including friction at the edges allows the configuration which we have discussed earlier, to be deformed, to a limited extent.

Our arguments do not apply to case (b) in *Figure* 1 because the rubber band takes a sharp bend even when it is on a single face, because it experiences a force from the other part of the same band which is looped around it. One can analyze this situation, again assuming a configuration of minimum length. It is an interesting exercise to verify (as already suggested by *Figure* 1(b)) that the geometry of the bands is the same as if there were two straight lines crossing each other! So the situation in *Figure* 1(b) is equivalent to a configuration which is realized by using two rubber bands, one along b and h and the other along l and h. (This is also sometimes used by shopkeepers.)

It will be interesting to also explore closed geodesics which wind around the cuboid more than once. These additional twists of a deceptively simple question have indeed opened up a Pandora's box of new possibilities to explore!

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Acknowledgement

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Suggested Reading

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[2] Patrick Honner, The Crooked Geometry of Round Trips, *Quantamagazine*, 2021, https://www.quantamagazine.org/the-crooked-geometry-of-round-trips-20210113/

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