Higher Secondary

Problem Corner

Computation of a Surd

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Problem. Compute the value of the following surd:

$$\left(8+3\sqrt{21}\right)^{1/3}+\left(8-3\sqrt{21}\right)^{1/3},$$

only real values being considered.

Solution. All the quantities involved are irrational, yet if we do the computations using a calculator, we find something remarkable:

$$\begin{aligned} 8+3\sqrt{21} &\approx 21.74772708,\\ \left(8+3\sqrt{21}\right)^{1/3} &\approx 2.791287847,\\ 8-3\sqrt{21} &\approx -5.747727084,\\ \left(8-3\sqrt{21}\right)^{1/3} &\approx -1.791287847, \end{aligned}$$

and:

$$\left(8+3\sqrt{21}\right)^{1/3}+\left(8-3\sqrt{21}\right)^{1/3}=1.000\ldots.$$

Well! The answer appears to be 1. Is it *exactly* equal to 1? — or so close that we cannot tell the difference? We shall show that the answer is exactly 1.

Let $u=\left(8+3\sqrt{21}\right)^{1/3}$ and $v=\left(8-3\sqrt{21}\right)^{1/3}$. We then have:

$$u^3+v^3=\left(8+3\sqrt{21}\right)+\left(8-3\sqrt{21}\right)=16,$$

and

$$\begin{aligned} uv &= \left(8+3\sqrt{21}\right)^{1/3}\cdot\left(8-3\sqrt{21}\right)^{1/3}\\ &= (64-189)^{1/3}=(-125)^{1/3}=-5. \end{aligned}$$

Keywords: Irrational numbers, cube roots, quadratic, discriminant, exact value

Next, we have:

$$u^3+v^3 = (u+v)\cdot\left(u^2-uv+v^2\right)$$
$$= (u+v)\cdot\left((u+v)^2-3uv\right).$$

Let $x=u+v$; we need to find x. (Note that x is a real number.) Since $u^3+v^3=16$ and $uv=-5$, we obtain

$$16 = x\left(x^2+15\right).$$

Hence x is a root of the cubic equation

$$x^3+15x-16=0.$$

Using the factor theorem, we are able to factorize the cubic; we obtain:

$$(x-1)\cdot\left(x^2+x+16\right)=0.$$

The only real root of this equation is 1 (the quadratic component yields non-real roots, as its discriminant is $1^2-(4\times 1\times 16)=-63$, which is negative). Hence $x=1$.

So the value of the given expression is 1. $\square$

A direct approach. Might there be another approach to solving this problem? Could we actually evaluate the two cube roots and thereby find their sum?

Let us assume that $\left(8+3\sqrt{21}\right)^{1/3}$ can be expressed in the form $a+b\sqrt{21}$, where a, b are rational numbers, and let us try to find a, b under this assumption. Taking the cubes of both the quantities, we obtain:

$$8+3\sqrt{21} = a^3+3\sqrt{21}a^2b+63ab^2+21\sqrt{21}b^3,$$

from which it follows, by equating the rational and irrational parts on both sides,

$$a^3+63ab^2=8,$$
$$a^2b+7b^3=1.$$

The equations imply that $a\neq 0$, $b\neq 0$. Let $k=b/a$; then, by assumption, k is a rational number. Substituting, we get:

$$a^3\left(1+63k^2\right)=8,$$
$$a^3\left(k+7k^3\right)=1.$$

These equations imply that $1+63k^2=8\left(k+7k^3\right)$, i.e.,

$$56k^3-63k^2+8k-1=0.$$

The cubic expression on the left is readily factorized, as the factor theorem tells us that $k-1$ is a factor. We thus obtain:

$$(k-1)\cdot\left(56k^2-7k+1\right)=0.$$

The quadratic component has discriminant $7^2-(4\times 1\times 56)=-175$, which is negative; so it does not yield any real roots (and therefore no rational roots either). The only real root is $k=1$, a rational number. It follows that $a=b$.

From this, it follows that $8a^3=1$, and hence that

$$a=\frac{1}{2},\qquad b=\frac{1}{2}.$$

Therefore, the real cube root of $8+3\sqrt{21}$ is

$$\frac{1}{2}+\frac{\sqrt{21}}{2},$$

and the real cube root of $8-3\sqrt{21}$ is

$$\frac{1}{2}-\frac{\sqrt{21}}{2},$$

implying that

$$\left(8+3\sqrt{21}\right)^{1/3}+\left(8-3\sqrt{21}\right)^{1/3}=1,$$

in agreement with what we had obtained earlier.

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