Problem Corner

Secondary

A Problem Related to an Equilateral Triangle

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In this article, we offer a geometric solution to a well-known problem which is generally solved using trigonometry.

The following problem can be found in the literature [1]; people generally use trigonometry to solve it. The statement of the problem and a typical solution are given below.

Problem. *If P is any point in the plane of an equilateral triangle ABC with side a and $PA = x$, $PB = y$ and $PC = z$, show that*

$$a^4 + x^4 + y^4 + z^4 = a^2\left(x^2 + y^2 + z^2\right) + x^2y^2 + y^2z^2 + z^2x^2.$$

Solution. The following solution is based on trigonometry.

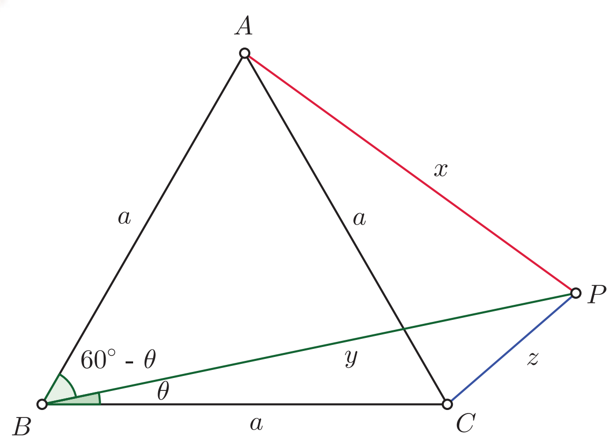

Figure 1.

Let $\measuredangle PBC = \theta$. Since $\triangle ABC$ is equilateral we have $\measuredangle ABP = 60^\circ - \theta$.

Keywords: Equilateral triangle, cosine rule, rotation, Stewart's theorem

From $\triangle BCP$ we get,

$$\cos\theta = \frac{a^2+y^2-z^2}{2ay}. \tag{1}$$

From $\triangle ABP$ we get,

$$\cos(60^\circ-\theta) = \frac{a^2+y^2-x^2}{2ay} \tag{2}$$

From (1) and (2) we obtain,

$$\sin\theta = \frac{a^2+y^2-2x^2+z^2}{2\sqrt{3}ay} \tag{3}$$

Eliminating θ from (1) and (3), we get,

$$\left(\frac{a^2+y^2-z^2}{2ay}\right)^2 + \left(\frac{a^2+y^2-2x^2+z^2}{2\sqrt{3}ay}\right)^2 = 1. \tag{4}$$

On simplification this gives ,

$$a^4+x^4+y^4+z^4 = a^2(x^2+y^2+z^2)+x^2y^2+y^2z^2+z^2x^2. \tag{5}$$

Note that (5) can also be written as

$$3\left(a^4+x^4+y^4+z^4\right) = \left(a^2+x^2+y^2+z^2\right)^2. \tag{6}$$

Can we solve the problem geometrically?

The answer is Yes! However, the task is quite challenging. We will investigate a few possible cases depending on the location of P, i.e., on whether P is inside or outside or on the triangle.

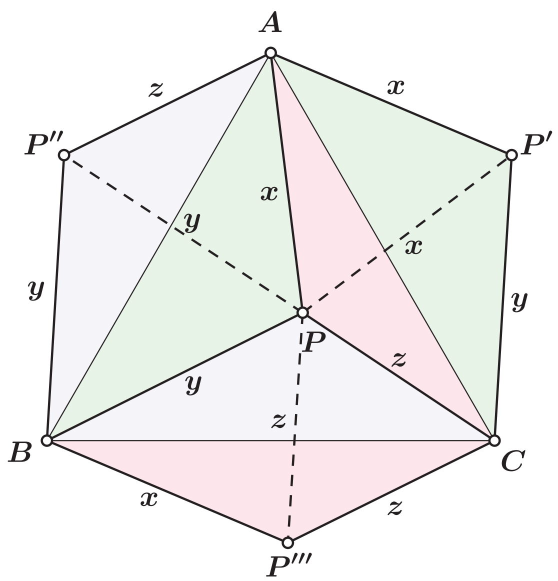

Figure 2.

Case 1: When P is inside the triangle. Let P be a point inside the triangle such that $PA = x$, $PB = y$ and $PC = z$. Now we will perform some rotation operations (see Figure 2).

- Rotate $\triangle ABP$ anticlockwise through an angle of 60° with respect to point A. Let P move to P'; then $\triangle ABP \cong \triangle ACP'$.
- Rotate $\triangle BCP$ anticlockwise through an angle of 60° with respect to point B. Let P move to P''; then $\triangle BCP \cong \triangle BAP''$.
- Rotate $\triangle CAP$ anticlockwise through an angle of 60° with respect to point C. Let P move to P'''; then $\triangle CAP \cong \triangle CBP'''$.

Observe that $\triangle APP'$, $\triangle BPP''$, $\triangle CPP'''$ are equilateral with side lengths x, y, z respectively. Also, $\triangle APP''$, $\triangle BPP'''$ and $\triangle CPP'$ have side lengths x, y, z. Note also that the area of hexagon $AP''BP'''CP'$ is twice the area of $\triangle ABC$.

Let each side of $\triangle ABC$ be a; let $2s = x + y + z$. Then:

$$2 \times \triangle ABC = (\triangle APP' + \triangle BPP'' + \triangle CPP''') + (\triangle APP'' + \triangle BPP''' + \triangle CPP')$$

$$\implies 2 \cdot \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4} x^2 + \frac{\sqrt{3}}{4} y^2 + \frac{\sqrt{3}}{4} z^2 + 3 \cdot \sqrt{s(s-x)(s-y)(s-z)}$$

$$\implies a^2 = \frac{1}{2}(x^2 + y^2 + z^2) + 2\sqrt{3} \cdot \sqrt{s(s-x)(s-y)(s-z)}$$

$$\implies \left(2a^2 - (x^2 + y^2 + z^2)\right)^2 = 3\left(2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4\right).$$

Simplifying we get,

$$a^4 + x^4 + y^4 + z^4 = a^2(x^2 + y^2 + z^2) + x^2y^2 + y^2z^2 + z^2x^2 \tag{7}$$

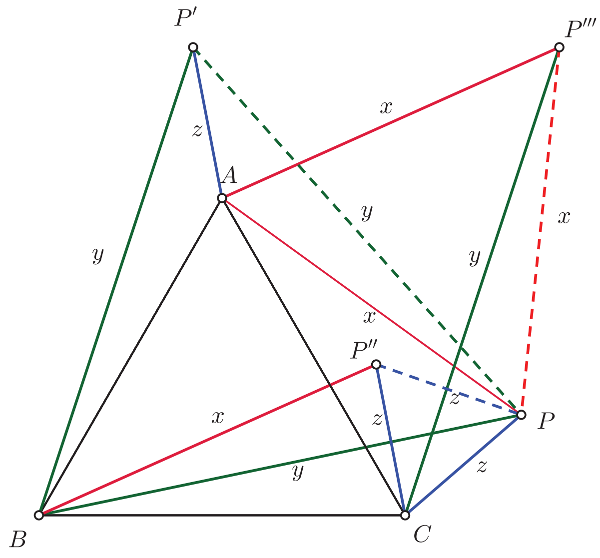

Figure 3.

Case 2: When P is outside the triangle.

- Rotate $\triangle BCP$ anticlockwise through an angle of 60° with respect to the point B. Let P move to P'; then $\triangle BAP' \cong \triangle BCP$.
- Rotate $\triangle ACP$ anticlockwise through an angle of 60° with respect to the point C. Let P move to P''; then $\triangle BCP'' \cong \triangle ACP$.
- Rotate $\triangle ABP$ anticlockwise through an angle of 60° with respect to the point A. Let P move to P'''; then $\triangle ACP''' \cong \triangle ABP$.

Let X denote the area of $\triangle BCP$ (equivalently, of $\triangle BAP'$); let Y denote the area of $\triangle ACP$ (equivalently, of $\triangle BCP''$); and let Z denote the area of $\triangle ABP$ (equivalently, of $\triangle ACP'''$). Observe that

$$\triangle ABC = \triangle ABP + \triangle BCP - \triangle ACP = Z + X - Y. \tag{8}$$

Also note that $\triangle APP'''$, $\triangle BPP'$, $\triangle CPP''$ are equilateral with side lengths x, y, z respectively; and that $\triangle APP'$, $\triangle BPP''$, $\triangle CPP'''$ have side lengths x, y, z. Let k denote the area of $\triangle APP'$; then $\triangle BPP''$ and $\triangle CPP'''$ also have area k.

So we have,

$$\triangle BCP + \triangle BPP'' = \triangle BCP'' + \triangle CPP'' \implies X + k = Y + \frac{\sqrt{3}}{4} z^2$$

$$\triangle ACP''' + \triangle CPP''' = \triangle ACP + \triangle APP''' \implies Z + k = Y + \frac{\sqrt{3}}{4} x^2$$

$$\triangle ABP' + \triangle APP' + \triangle ABP = \triangle BPP' \implies X + k + Z = \frac{\sqrt{3}}{4} y^2$$

Adding the above three equalities, we get,

$$\begin{aligned}
& 2X + 2Z + 3k = 2Y + \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2) \\
\implies & 2(X + Z - Y) = \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2) - 3k \\
\implies & 2\triangle ABC = \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2) - 3\sqrt{s(s-x)(s-y)(s-z)} \\
\implies & 2 \times \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2) - 3\sqrt{s(s-x)(s-y)(s-z)} \\
\implies & a^2 = \frac{1}{2}(x^2 + y^2 + z^2) - 2\sqrt{3}\sqrt{s(s-x)(s-y)(s-z)} \\
\implies & \left\{2a^2 - (x^2 + y^2 + z^2)\right\}^2 = 3(2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4).
\end{aligned}$$

Simplifying, we get,

$$a^4 + x^4 + y^4 + z^4 = a^2(x^2 + y^2 + z^2) + x^2y^2 + y^2z^2 + z^2x^2.$$

Case 3: When P lies on a side of the triangle. This special case can easily be tackled by Stewart's theorem [3]. (A statement of the theorem is given in Box 1 at the end of the article.)

Suppose that P lies on side BC. Then $a = y + z$. In this case the relation to be proved reduces to $x^2 = y^2 + yz + z^2$ or equivalently, $a^2 = x^2 + yz$.

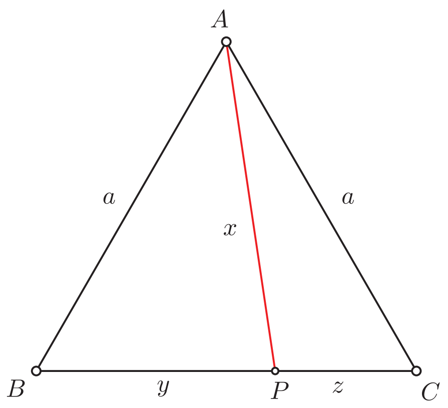

Figure 4.

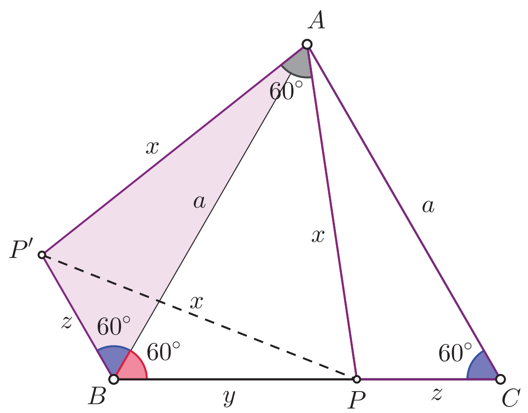

Figure 5.

Applying Stewart's theorem in $\triangle ABC$ (Figure 4) we get,

$$\begin{aligned} a^2y + a^2z = a(yz + x^2) &\implies a(y+z) = x^2 + yz \\ &\implies a^2 = x^2 + yz \qquad [\text{since } a = y+z] \\ &\implies (y+z)^2 = x^2 + yz \implies y^2 + yz + z^2 = x^2. \end{aligned}$$

Proof using rotation. The above relation can also be deduced from Figure 5.

If we rotate $\triangle APC$ clockwise through an angle of 60° with respect to A, then C goes to B and P goes to P' and $\triangle APC \cong \triangle ABP'$. Hence $\measuredangle PBP' = 60^\circ + 60^\circ = 120^\circ$. Also $\measuredangle PAP' = 60^\circ$ and $AP = AP'$. Hence $\triangle APP'$ is equilateral, so $PP' = x$. Now from $\triangle BPP'$ we easily get, $x^2 = y^2 + yz + z^2$. (We invite the reader to find a proof without the use of the cosine rule.)

A different interpretation

It is interesting to note that the relation to be proved can be interpreted differently. If we let x, y, z be three given distances from a fixed point and treat a as variable, then we get an interesting outcome: two possible equilateral triangles. If the side lengths of these two equilateral triangles are a_1 and a_2 then solving (6) as a quadratic in a^2 we get,

$$a_1^2 = \frac{1}{2}(x^2 + y^2 + z^2) + 2\sqrt{3}\cdot\sqrt{s(s-x)(s-y)(s-z)},$$
$$a_2^2 = \frac{1}{2}(x^2 + y^2 + z^2) - 2\sqrt{3}\cdot\sqrt{s(s-x)(s-y)(s-z)},$$

which give

$$a_1^2 + a_2^2 = x^2 + y^2 + z^2 \tag{9}$$

$$\implies \frac{\sqrt{3}}{4}a_1^2 + \frac{\sqrt{3}}{4}a_2^2 = \frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}y^2 + \frac{\sqrt{3}}{4}z^2. \tag{10}$$

Hence we can state the following result.

The sum of areas of two equilateral triangles each of which has its vertices at three given distances from a fixed point is equal to the sum of the areas of the equilateral triangles described on these distances.

An equivalent problem. The following problem, due to Francisco Javier García Capitán [2] of Spain, provides an equivalent interpretation to the above problem: *Given three concentric circles with radii r_1, r_2, r_3, find the lengths of the sides of equilateral triangles with one vertex on each circle.*

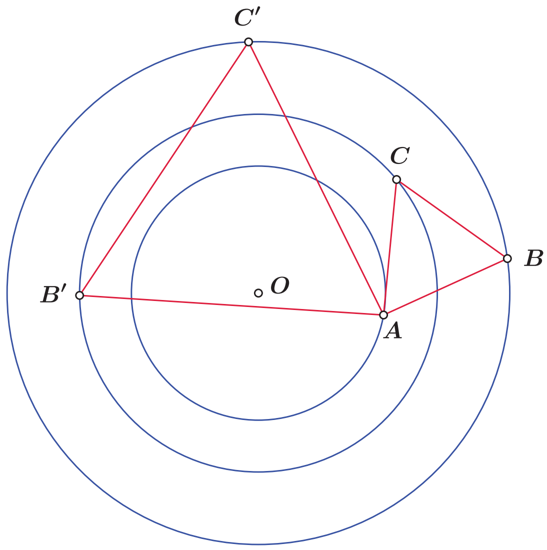

Figure 6.

Figure 6 depicts the problem. It turns out that if L is the length of a side of any such equilateral triangle, then

$$L^4 - \left(r_1^2 + r_2^2 + r_3^2\right) L^2 + r_1^4 + r_2^4 + r_3^4 - r_1^2 r_2^2 - r_2^2 r_3^2 - r_3^2 r_1^2 = 0.$$

References

1. Gardner, Martin, *Mathematical Circus*, MAA, 1992
2. Capitán, Francisco Javier García, Problemas y Soluciones (Volumen 1), Priego de Córdoba, España, Abril de 2020
3. Wikipedia, “Stewart’s theorem” from https://en.wikipedia.org/wiki/Stewart%27s_theorem

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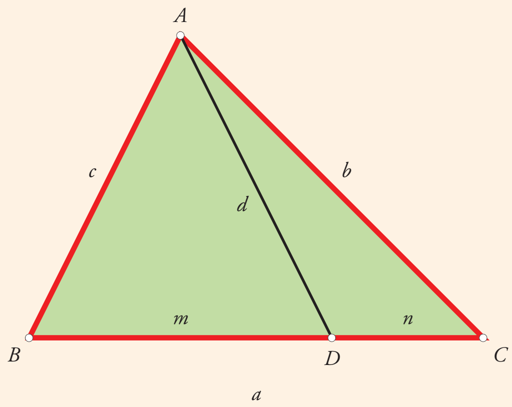
Statement of Stewart's theorem

Let ABC be any triangle, and let D be any point on side BC. Let a, b, c be the lengths of BC, CA, AB, respectively, and let d be the length of AD. Finally, let $BD : DC = m : n$. (See the figure below. Note that m, n could also represent the lengths of BD, DC, respectively.)



Then Stewart's theorem [3] states that

$$a\left(d^2 + mn\right) = b^2 m + c^2 n.$$

Box 1. Stewart's theorem