Inequalities - Part 1

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Inequalities are encountered in almost every branch of mathematics. They have fascinating properties and many applications. In this series of articles, we will consider how different forms may be reduced or converted into known inequalities. We will study various problems taken from mathematical Olympiads across the world. The motivation and structure of the text is due to the wonderful resources [1], [5] and [2]. In the first part of the article, we study the most fundamental and basic inequality of all—the arithmetic mean-geometric mean inequality.

The AM-GM Inequality

As already noted, we start with the AM-GM inequality. It is regarded as one of the most fundamental and basic inequalities of all, as it implies so many other results. The simplest form of the AM-GM inequality is:

Theorem 1 (AM-GM inequality)**.** *For positive real numbers a, b, the following inequality holds:*

$$\frac{a+b}{2} \geq \sqrt{ab}. \tag{1}$$

Moreover, equality holds if and only if $a = b$.

Proof. As all the quantities involved are positive, the inequality is equivalent to the one obtained by squaring it. That is, it is equivalent to $(a+b)^2 \geq 4ab$. This in turn is equivalent to $(a+b)^2 - 4ab \geq 0$, i.e., to $(a-b)^2 \geq 0$. The last statement is clearly true; hence the AM-GM inequality follows. Moreover, equality holds in the last statement precisely when $a = b$. Hence equality holds in the AM-GM inequality precisely when $a = b$. □

The inequality can be further generalized in the form below.

Theorem 2 (AM-GM inequality for n numbers)**.** *For any n positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds true:*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}. \tag{2}$$

Proof. There are many ways to prove this inequality. Most often, it is proved by induction on n, starting with the case $n = 2$. We present a different and very elegant proof using the exponential function; it was first given by Pólya. The proof goes as follows.

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We consider the function $g(x) = e^{x-1} - x$. Its first and second derivatives are $g'(x) = e^{x-1} - 1$ and $g''(x) = e^{x-1}$. Observe that $g(1) = 0$ and $g'(1) = 0$. The second derivative is always positive, hence g is a strictly convex function with an absolute minimum value of 0 at $x = 1$. It therefore follows that:

$$x \leq e^{x-1} \quad \text{for all real values of } x. \tag{3}$$

Note that equality holds precisely when $x = 1$.

Now consider a collection of n real non-negative numbers $a_1, a_2, \ldots, a_n$. Let b be their arithmetic mean. Then we have, using the above result:

$$\begin{aligned}
\frac{a_1}{b} \cdot \frac{a_2}{b} \cdots \frac{a_n}{b} &\leq e^{a_1/b-1} \cdot e^{a_2/b-1} \cdots e^{a_n/b-1} \\
&= e^{(a_1+a_2+\cdots+a_n)/b-n} = e^{n-n} = 1, \\
\therefore\ a_1 \cdot a_2 \cdots a_n &\leq b^n, \\
\therefore\ (a_1 \cdot a_2 \cdots a_n)^{1/n} &\leq b,
\end{aligned} \tag{4}$$

which proves the AM-GM inequality. Moreover, equality will hold precisely when all of the values a_1/b, $a_2/b, \ldots, a_n/b$ are equal to 1, which means that the n numbers $a_1, a_2, \ldots, a_n$ are all equal to each other. □

Weighted form of AM-GM inequality. It is possible to generalise the AM-GM inequality still further, into a weighted form. Let $a_1, a_2, \ldots, a_n$ be n real positive numbers, and let $w_1, w_2, \ldots, w_n$ be n positive weights with sum 1. The weighted form of the AM-GM inequality is the following statement:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}. \tag{5}$$

The particular case when all the weights are equal to $1/n$ is Theorem 2. The proof runs along exactly the same lines as the one given above. For details, please see [4].

Some applications of the AM-GM inequality

We now look at some inequalities that make use of the AM-GM inequality.

Example 1: Nesbitt's Inequality. This states that for any three positive real numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \tag{6}$$

Proof. The solution comes from [1] and [2].

Let k denote the quantity on the left side. Then we have:

$$k = \frac{1}{2}\Big((a+b) + (b+c) + (c+a)\Big) \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} - 6\right). \tag{7}$$

Let us now substitute new variables $x = a+b$, $y = b+c$, $z = c+a$ in (7). Then:

$$\begin{aligned}
k &= \frac{1}{2}(x+y+z) \cdot \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 3 \\
&= \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}\right) - \frac{3}{2}.
\end{aligned}$$

Now for any positive real number a, we have

$$a + \frac{1}{a} \geq 2,$$

by the AM-GM inequality. Therefore we can say that

$$k \geq \frac{1}{2}(2+2+2) - \frac{3}{2}, \quad \text{i.e.,} \quad k \geq \frac{3}{2}.$$

This proves the stated inequality. □

Example 2. For any three positive real numbers a, b, c,

$$\left(a^2 b + b^2 c + c^2 a\right) \cdot \left(b^2 a + c^2 b + a^2 c\right) \geq 9a^2 b^2 c^2. \tag{8}$$

Proof. The solution comes from [2]. We apply the AM-GM inequality to the bracketed terms on the left side:

$$\begin{aligned} a^2 b + b^2 c + c^2 a &\geq 3\left(a^3 b^3 c^3\right)^{1/3} = 3abc, \\ b^2 a + c^2 b + a^2 c &\geq 3\left(a^3 b^3 c^3\right)^{1/3} = 3abc. \end{aligned}$$

Therefore by multiplication we get:

$$\left(a^2 b + b^2 c + c^2 a\right) \cdot \left(b^2 a + c^2 b + a^2 c\right) \geq 9a^2 b^2 c^2.$$

Example 3: A problem from IMO 1964. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc. \tag{9}$$

Proof. The solution comes from [3]. We use the substitution

$$a = y + z, \quad b = z + x, \quad c = x + y.$$

If we solve for x, y, z in terms of a, b, c we get:

$$x = \frac{b+c-a}{2}, \quad y = \frac{c+a-b}{2}, \quad z = \frac{a+b-c}{2}.$$

Therefore, by the triangle inequality, x, y, z are positive numbers. We now rewrite the inequality in the following equivalent form:

$$2x(y+z)^2 + 2y(z+x)^2 + 2z(x+y)^2 \leq 3(x+y)(y+z)(z+x). \tag{10}$$

On expanding the squared terms and simplifying, this assumes the following equivalent form:

$$x^2 y + y^2 z + z^2 x + x^2 z + y^2 x + z^2 y \geq 6xyz.$$

But this easily follows from the AM-GM inequality:

$$\begin{aligned} & x^2 y + y^2 z + z^2 x + x^2 z + y^2 x + z^2 y \geq 6\left(x^2 y \cdot y^2 z \cdot z^2 x \cdot x^2 z \cdot y^2 x \cdot z^2 y\right)^{1/6}, \\ \text{i.e.,}\ & x^2 y + y^2 z + z^2 x + x^2 z + y^2 x + z^2 y \geq 6xyz. \end{aligned}$$

Example 4. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \leq 1. \tag{11}$$

Proof. The solution comes from Nguyen Manh Dung in [1].

We shall use the weighted AM-GM this time.

The following three inequalities all follow from the weighted AM-GM inequality:

$$\frac{a^2+b^2+c^2}{a+b+c} \geq \left(a^a b^b c^c\right)^{1/(a+b+c)},$$
$$\therefore\ a^2+b^2+c^2 \geq a^a b^b c^c \quad (\text{since } a+b+c=1).$$

Next:

$$\frac{ab+bc+ca}{a+b+c} \geq \left(a^b b^c c^a\right)^{1/(a+b+c)},$$
$$\therefore\ ab+bc+ca \geq a^b b^c c^a \quad (\text{since } a+b+c=1),$$

and:

$$\frac{ac+ba+cb}{a+b+c} \geq \left(a^c b^a c^b\right)^{1/(a+b+c)},$$
$$\therefore\ ac+ba+cb \geq a^c b^a c^b \quad (\text{since } a+b+c=1).$$

Summing the three inequalities we get

$$(a+b+c)^2 \geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b,$$
$$\therefore\ 1 \geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b,$$

as required. $\square$

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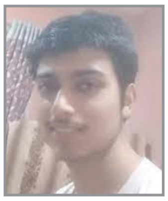

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