Solution to a Functional Equation

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Functional equations occur quite frequently in problem contests. Typically they specify certain properties of an unknown function, on the basis of which we are supposed to find that function. If the answer is unique, it means that those properties characterise that particular function. We explore one such problem in this article.

Problem. To find all continuous functions f defined on the set of real numbers and taking real values, with the following two properties:

- $f(0) = 1$;
- $f(u+v+1) = f(u) + f(v)$ for all real numbers u, v.

By simple experimentation, we find that the function $f(x) = x + 1$ satisfies all the given conditions. Could it be the only solution? Let us explore. We consider various classes of numbers.

Case 1: x is a positive integer. We make repeated use of the property $f(u+v+1) = f(u) + f(v)$ for all real numbers u, v. The substitution $u = 0, v = 0$ gives:

$$f(1) = f(0+0+1) = f(0) + f(0) = 1 + 1 = 2.$$

We see that $f(1) = 1 + 1$. Next, the substitution $u = 1, v = 0$ gives:

$$f(2) = f(1+0+1) = f(1) + f(0) = 2 + 1 = 3.$$

We see that $f(2) = 2 + 1$.

Now assume that $f(k) = k + 1$ for some positive integer k. The substitution $u = k, v = 0$ gives:

$$\begin{aligned} f(k+1) &= f(k+0+1) = f(k) + f(0) \\ &= (k+1) + 1 = k + 2. \end{aligned}$$

Using the principle of induction, it follows that $f(x) = x + 1$ for every positive integer x.

Case 2: x is a negative integer. Let x be a negative integer, and let $y = -x$. Since y is a positive integer, we have $f(y) = y + 1$. The substitution $u = x, v = y$ gives:

$$\begin{aligned} f(1) &= f(x+y+1) = f(x) + f(y) \\ &= f(x) + (y+1) = f(x) - x + 1, \end{aligned}$$

hence $2 = f(x) - x + 1$, i.e., $f(x) = x + 1$. Therefore $f(x) = x + 1$ for every negative integer x.

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Case 3: x is a non-integral rational number. We first show that for any rational number x, the following are true:

(1) $f(x+1) = f(x) + 1$. Equivalently: $f(x-1) = f(x) - 1$.

(2) $f(-x) = 2 - f(x)$.

(3) For any positive integer m,
$$f(mx) = mf(x) - (m-1).$$

We justify these as follows.

$$\begin{aligned} f(x+1) &= f(x+0+1) = f(x) + f(0) \\ &= f(x) + 1. \end{aligned}$$

Next:

$$2 = f(x + (-x) + 1) = f(x) + f(-x),$$
$$\therefore\ f(-x) = 2 - f(x).$$

Finally, the relation $f(mx) = mf(x) - (m-1)$ is certainly true for $m = 1$:

$$f(x) = 1 \cdot f(x) - (1-1).$$

Assume that the relation $f(mx) = mf(x) - (m-1)$ is true for $m = k$, where k is some positive integer. Then

$$f(kx) = kf(x) - (k-1).$$

Therefore:

$$\begin{aligned} f((k+1)x) &= f(kx + x) \\ &= f(kx + (x-1) + 1) \\ &= f(kx) + f(x-1) \\ &= kf(x) - k + 1 + f(x) - 1 \\ &= (k+1)f(x) - k \\ &= (k+1)f(x) - (k+1-1). \end{aligned}$$

By the principle of induction, we conclude that $f(mx) = mf(x) - m + 1$ for all positive integers m.

Now let $x = p/q$ where p and q are integers, $q > 0$. Then $qx = p$ is an integer, therefore

$$f(qx) = qx + 1.$$

But $f(qx) = qf(x) - q + 1$. Hence:

$$qf(x) - q + 1 = qx + 1.$$

Solving for $f(x)$, we obtain:

$$f(x) = x + 1.$$

It follows that $f(x) = x + 1$ for all rational numbers x.

Case 4: x is a real but irrational number. Now let x be a real number. Then there exists a sequence $x_1, x_2, x_3, \ldots$ of rational numbers such that $\lim_{n\to\infty} x_n = x$. Since (by supposition) f is a continuous function,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

But $f(x_n) = x_n + 1$ for all n. It follows that

$$\lim_{n\to\infty} f(x_n) = x + 1.$$

Therefore $f(x) = x + 1$ for all real numbers x.

Conclusion. There is precisely one function f satisfying the given conditions: $f(x) = x + 1$ for all real numbers x.

References

1. Shuborno Das, "Functional Equations", *At Right Angles*, Vol 7, No. 2, July 2018, from https://azimpremjiuniversity.edu.in/SitePages/resources-ara-vol-7-no-2-july-2018-functional-equations-part-1.aspx

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