An Olympiad Challenge

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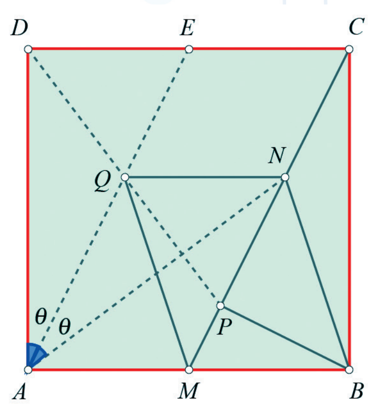
Presenting a solution to a problem from Crux Mathematicorum

Problem

In a square $ABCD$, let M be the midpoint of AB, let P be the projection of point B onto line CM, and let N be the midpoint of segment CP. Let the angle bisector of $\measuredangle DAN$ intersect line DP at point Q. Prove that quadrilateral $BMQN$ is a parallelogram.



$A = (0,0)$

$B = (10,0)$

$C = (10,10)$

$D = (0,10)$

$M = (5,0)$

Figure 1

Solution

We use an approach combining methods of coordinate geometry and trigonometry. Assign coordinates so that A lies at the origin, and the sides AB and AD lie along the x-axis and the y-axis respectively. Choose the scale so that the side of the square is 10

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units (this is merely to avoid fractions). Then the coordinates of A, B, C, D, M are as shown in the figure. Let AQ extended meet side CD at E. Let $\measuredangle DAE = \theta = \measuredangle EAN$.

The coordinates of P may be found by using similarity (triangles MPB, BPC and MBC are similar, so $MP/PB = 1/2 = BP/PC$, which means that $MP/PC = 1/4$; so P divides MC in the ratio 1 : 4); or by solving a pair of simultaneous equations (the equation of CM is $y = 2(x - 5)$, and the equation of BP is $y = -(x - 10)/2$; solving these simultaneously we get the coordinates of P). Using either approach we get $P = (6, 2)$ and hence $N = (8, 6)$. Therefore the slope of AN is 3/4.

Since $\measuredangle NAB = 90^\circ - 2\theta$, we get $\tan(90^\circ - 2\theta) = 3/4$, hence:

$$\tan 2\theta = \frac{4}{3}, \quad \therefore \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3}.$$

Solving the resulting quadratic equation for tanθ, we find that

$$\tan\theta = \frac{1}{2}.$$

This means that the equation of AE is $y = 2x$. Also, the equation of DP is

$$y - 10 = \frac{10-2}{0-6}(x-0), \quad \text{i.e., } y = -\frac{4}{3}x + 10.$$

Solving the two equations simultaneously, we get $Q = (3, 6)$. We see that QN is parallel to the x-axis and has length 5 units. That is, QN is equal and parallel to AM and to MB. It follows that quadrilateral $AQNM$ is a parallelogram, and so is quadrilateral $BMQN$. □

RAKSHITHA is a math enthusiast studying in class 12 at The Learning Centre, PU College, Mangalore. She has a deep interest in Analysis, Number Theory and Combinatorics. Other than Mathematics, she is fascinated by the sciences and wishes to pursue pure research in the future. She participated in the Indian National Mathematics Olympiad in 2019 and 2020. She may be contactedat krraja1974@rediffmail.com