Features

On some Geometric Constructions in the Sulvasutras from a Pedagogical Perspective – II

S.G. DANI
MEDHA LIMAYE

In the first part of this article we described briefly the setting of the sulvasutra geometry and construction of various basic rectilinear figures with a given area (or equivalently transformation of shapes into one another, with the same area). In this sequel we continue on the topic, branching out along the following themes: Firstly, using some arithmetic, we discuss conversion of multiple squares together into one, more efficiently than by simple repeated augmentation of squares as described in part I. In the second section we discuss the topic at hand with regard to the semicircles and circles. The last section is devoted to discussion of certain constructions which are not found explicitly in the sulvasutras, but could have been the basis of some of the knowledge that is propounded in them, specifically, the Pythagoras theorem and the value of $\sqrt{2}$.

I. Merging multiple squares into one

Merging a number of squares into one, irrespective of their sizes, can be done geometrically, in principle, by successively following the method described in §2.1 in part I of this article. In this section we discuss some further points on the theme. The first two subsections below concern possible simplifications in adjoining a number of squares of the same size. For simplicity of exposition we adopt here the modern

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symbolic notation, and refer to n squares of unit size, etc., though such symbolism is not found in the sulvasutras, and all statements are expressed in words. In the last subsection we discuss a specific way in which the process for joining squares was used.

I.1. Combining n squares through diagonal constructions. When $n = k^2 + l^2$, where k and l are two natural numbers, a square with area n can be constructed as the square on the diagonal of a rectangle with sides k and l units respectively.

Such an application is seen explicitly in Katyayana sulvasutra[1], in KSS 2.4 (2.8) (see the footnote below[2]), where a square with area 10 units is prescribed to be constructed as the diagonal of a rectangle with sides being 3 units and 1 unit.

More generally for n expressed as a sum of r square integers, a square with area n units can be produced geometrically through $r - 1$ iterations of the above procedure, as against $n - 1$ iterations if one follows only a simple-minded procedure of adding a unit square at a time. The process works irrespective of the ordering of the r squares, but a simple choice would be to take the first term to be the largest square not exceeding n, the next to be the largest square not exceeding the balance to be added, etc.; thus, for example, 15 would be expressed as $3^2 + 2^2 + 1 + 1$, and 59 as $7^2 + 3^2 + 1$. Incidentally, for constructing a square with area 15 units one could also use alternatively the fact that $15 = 4^2 - 1$ and use the process for the difference of two squares; in the same way, for 29 one could use $29 = 7^2 - 4^2 - 2^2$.

I.2. Combining n squares via altitudes of isosceles triangles. Katyayana sulvasutra describes an alternative method for producing a square with area n. KSS 6.7 (6.7) states the following:

> When it is desired to put together a number of squares (of unit size) construct an isosceles triangle whose base is one less than the number, and the other two (equal) sides add up to one more than the number; the altitude of the triangle then serves as the side for the desired square.[3]

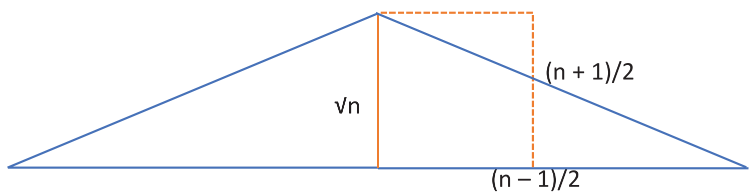

Figure 1. Construction of a square of area n, via isosceles triangles (for $n > 1$)

We note that for the given choices each of the slanted sides of the isosceles triangle is $\frac{1}{2}(n+1)$ and half the base is $\frac{1}{2}(n-1)$. Together with the altitude of the triangle these form a right angled triangle, and hence by the Pythagoras theorem the square of the altitude is $\frac{1}{4}(n+1)^2 - \frac{1}{4}(n-1)^2$. Thus the construction is based on (knowledge of) the identity

$$(n+1)^2 - (n-1)^2 = 4n.$$

1 As in part I we will not use diacritical marks to indicate pronunciation; a Glossary is included as a guide for pronunciation.

2 The first number is as per [7], and the parenthetical one corresponds to the numbering adopted in some of the older sources, including [5] and [6].

3 It may be of some interest to note here that the word used in the sutra for altitude is *iṣu*, which corresponds to "arrow", evidently derived from the analogy of an isosceles triangle with a stretched bow-string. Similar terminology occurs also in Jaina mathematics from the first millennium BCE; see [3].

The identity is straightforward to verify in terms of symbolic algebra. However, in their context it would have been derived differently, in absence of symbolic algebraic methods; one such possibility consistent with the ethos of the time is through counting of tiles (note that though for convenience we use algebraic notation here, the underlying statement can readily be formulated without it): in a grid of square tiles with $n+1$ rows and columns we see that there are $4n$ tiles along the edges, and together with the tiles in the middle square, which has $n-1$ rows and columns, they account for all tiles; hence $(n+1)^2-(n-1)^2=4n$.

I.3. Enhancement of units. The performance of certain yajnas was repeated periodically, and at each successive event the sizes of the vedis were to be increased in a stipulated manner; the first time the area would be $7\frac{1}{2}$ (square) units (the unit being *puruṣa*, measured as the height of a man, normally the yajamana, with uplifted arms), the next time it would be $8\frac{1}{2}$ units, and then $9\frac{1}{2}$ units, and so on, in the same shape (as scaled up figures). The shapes involved being quite intricate, it would be quite complicated to adjust each part; instead this was achieved by *enhancing the size of the unit* adopted for the construction, and following the same steps as before. Thus for instance for the first step as above where the net area is to be scaled up from $7\frac{1}{2}$ units to $8\frac{1}{2}$, namely by a factor of $1+\frac{2}{15}$ the measure of the unit length would be increased to $\sqrt{(1+\frac{2}{15})}$; this desired length for the new unit would be determined, geometrically, as the length of the side of the square obtained by joining to the unit square a square of size $\frac{2}{15}$; for the latter, one may start with a rectangle with sides $\frac{2}{5}$ and $\frac{1}{3}$, for example, and turn it into a square, by the procedure discussed earlier, in part I.

II. Conversions between circle, semicircle and square

There is a sutra in Manava sulvasutra (MSS 1.8 (10.1.1.8)) which concerns describing a circle, a semicircle and a square with the same area, they being the shapes of the three *agni*s (fireplaces), stipulated to have the same area. While the occurrence of the three of them together is unique to Manava sulvasutra, conversion from circle to square and vice versa is treated in all the four sulvasutras. The method for transforming a circle to semicircle is exact while the methods for inter-conversions between the circle and the square hold only approximately, as we shall see below.

II.1. Transforming a circle to a semicircle. According to the sutra mentioned above concerning this, to construct a semicircle with the same area as a given circle, the radius of the desired semicircle is to be taken as the size of the diagonal of a square whose side is the radius of the given circle; see Figure 2 for an illustration of the construction - a square is formed of two radial segments, OA and OB, perpendicular to each other, and its diagonal OP is used for the radius of the desired semicircle. The area of the circle with the new radius will then be twice that of the original circle, and hence the area of the semicircle is the same as the area of the given circle. The reader may observe the close connection of this with the problem of doubling of a square discussed in part 1 of this article.

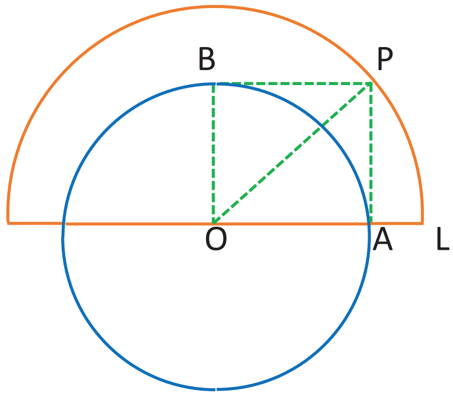

Figure 2. A semicircle drawn to have the same area as a given circle.

The description in the sutra does not specifically concern itself with the converse issue of starting with a semicircle and turning it into a circle. However, it is reasonable to suppose that if called upon to do so they would have retraced the steps: construct a square whose diagonal coincides with the radius of the given semicircle; the side of the desired square can be obtained by taking the endpoint of $\frac{1}{4}$th of the arc segment from one end, viz., LP as in Figure 2, and dropping the perpendicular on the diameter of the semicircle (to determine the point A).

II.2. Circling the square. The sutrakaras are also seen to have been interested in converting a square into a circle and the other way. The methods prescribed for this however happen to be approximate, and rather crude ones at that.

For converting a square to a circle (*maṇḍala*) BSS 2.9 (1.58) has the following instruction.

> Desiring to convert a square to a circle drop the semi-diagonal from the midpoint (of the square) along the line of symmetry perpendicular to a pair of sides, and draw a circle including one-third of the part remaining outside the square.

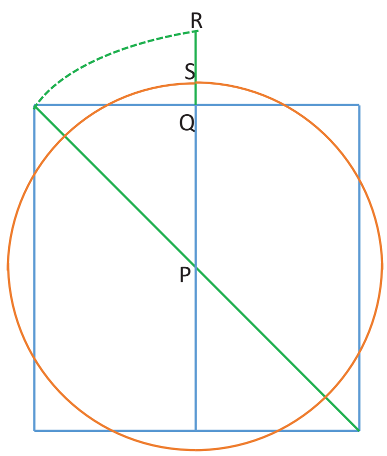

Figure 3. Circling the square

In place of “line of symmetry perpendicular to a pair of sides” that we have referred to, the sutra actually mentions the eastward direction, it being assumed (implicitly) that the sides of the squares are set along the cardinal directions. See Figure 3 for an illustration of the construction; here P is the midpoint of the square, PR is the semi-diagonal laid along a line of symmetry, Q is its point of intersection with the side, and S is the point on QR at a distance one-third the length of QR, from Q, which is prescribed as the point through which the desired circle is to pass.

For the unit square the area of the circle produced as above is seen to be $\frac{\pi}{4}(\frac{2+\sqrt{2}}{3})^2$ (with π as the area of the unit circle), in place of the desired value 1; this involves an error of about 1.7%. To get a sense about the error involved, it may be seen that for the value as above to be 1, corresponds to a value of π which is approximately 3.08, as against about 3.14, at 2 decimal places; it should be borne in mind however that it was not their purpose to determine the ratio π as we understand it, but only to produce a circle equivalent to the square in area to meet their requirement in the ritual context, and evidently the value was considered good enough for the purpose.[4]

Notwithstanding the degree of error the heuristic involved in the process is worth taking note of. Clearly the radius of the desired circle would have to be between half the side of the square and half the diagonal, and $\frac{1}{3}$rd of the extra part was added to the former in this light, presumably based on some visual intuition.[5] It turns out that if in place of “one third” of the jutting out part they were to add $\frac{3}{10}$th of the same, they would have got a much closer approximation for the area of the circle produced! However, though the latter may now seem a natural possibility to have been considered, in the context of the decimal place value system of representation of numbers, in India such representation remained confined to whole numbers and did not get extended to fractions until the modern era. At any rate, the fact that one had to look for an ad hoc choice, without any means of determining the factor mathematically, was a major handicap in dealing with the issue at that time.

II.3. Squaring the circle. The converse problem of finding a square equal in area to a given circle is also addressed.[6] However the resolution is not through a geometrical construction in the same spirit as seen so far. Instead, numerical values are assigned for the ratio of the side of the desired square to the diameter of the given circle. The relatively more accurate value given by Baudhayana in BSS 2.10 (I.59) (described in words in the original) is

$$\frac{7}{8}+\frac{1}{8\times 29}-\frac{1}{8\times 29\times 6}+\frac{1}{8\times 29\times 6\times 8}.$$

This involves the same degree of error as in the solution of circling the square as above, but in the opposite direction. This readily suggests that the ratio was obtained by inverting the ratio involved in the converse process of turning a square into a circle. How the computations went is unclear, but one thing that seems clear is that the fairly accurate formula for $\sqrt{2}$ as a fraction that they found (see the next section), was inspired by this problem; the value had no other ostensible purpose in their context, as the geometrical construction of $\sqrt{2}$ as *dvikaraṇī* would suffice for all their other requirements, and it would also be more accurate.

4 In [1] (followed up also in [2]) a case is made that one of the sutras in Manava sulvasutra gives an analogous construction with a better result, with an error less than $\frac{1}{2}$%. We shall however not go into the details here.

5 A preliminary option that would suggest itself in respect of such “interpolation” would have been to take the average, viz. a factor of $\frac{1}{2}$ in place of $\frac{1}{3}$, which they apparently ruled out as unsuitable.

6 It should be noted that the problem is considered here from a practical point of view and should not be confused with the Greek problem of “squaring the circle” with only a straight edge and compass (and calls for an exact solution), which was later realized to be impossible, on account of π being a transcendental number, as established by Ferdinand von Lindemann in 1882.

III. Other related constructions

As noted earlier the sulvasutras were composed with the objective of providing instruction related to construction of vedis and agnis, and did not concern the broader aspect of creating a repository of all knowledge of that time on any topic. In particular there are no proofs given for the principles enunciated or the constructions proposed for various purposes. While in case of many statements one can envision how the scholars of that time may have arrived at them, there are various instances where one would wonder about their reasoning, or the process by which they would have arrived at the conclusion in question. In this section we discuss two such instances, with surmises about how the conclusions would have been arrived at, via the knowledge or familiarity with constructions akin to those that we discussed so far.

III.1. A value for $\sqrt{2}$. BSS 2.12 (1.61) describes an approximate value for $\sqrt{2}$ which may be expressed as

$$1+\frac{1}{3}+\frac{1}{3\times 4}-\frac{1}{3\times 4\times 34};$$

the original is of course in words, and the form of presentation as above (rather than its simplified fractional form as $\frac{577}{408}$) conveys how the number is described, giving each successive term in the expression as a certain part of the preceding one. One may also expect that the way in which it is expressed may hold a clue as to how the formula was arrived at. The value as above turns out to be quite close to the actual value of $\sqrt{2}$, coinciding in its decimal expansion upto five decimal places, as $1.4142156\ldots$[7] in place of $1.4142139\ldots$.[8]

The numerical value for $\sqrt{2}$ would not have played a role in their constructions for practical purposes, since they could always construct the magnitude geometrically with much less effort. The endeavour of producing an expression as above for it in terms of fractions seems to have been inspired by the problem of squaring the circle, discussed in §II.3 above.

While it is not known how they arrived at the expression (see [1] for more discussion and references on this) one of the suggestions in this respect, due to Bibhutibhushan Datta [4] (see also [7]) would be worth recalling here, it being in the spirit of our present theme of conversion of figures, as treated in the sulvasutras. It may be described as follows.

The aim is to find the side of a square obtained by putting together two unit squares, in numerical terms, as against by a geometrical procedure, the latter being already described. Considering the problem as one of converting a rectangle with sides 2 and 1, the two unit squares being put side by side, the geometric procedure described earlier involves bisecting one of them and putting one part along the perpendicular side, so that we arrive at a figure which is a difference of two squares, which is subsequently adjusted for. Instead, we now divide the second unit square in three equal strips along a pair of parallel sides of the square. (The reader is advised to follow Figure 4 stepwise for the construction described here.)

7 It may be appropriate to recall here that the Babylonians also had a similarly accurate value, around 1700 BCE, in fact with a slightly smaller error, which is on the opposite side.

8 The closeness of the value has led some authors to argue that the sutrakaras had the idea of irrational numbers, and recognized $\sqrt{2}$ as one such. Such arguments however bear no substance. Evidently they would have arrived at the formula by an iterative procedure (whether the one discussed here or some other one), and at the stage of arriving at the expression it would be apparent that the actual value is not reached exactly, and as such the formula involves an error; the process involved may also hold a clue on the degree of error involved. Realization of these, though remarkable in the given context, has little to do with realizing the number to be irrational, namely that there is *no way at all* to write it as a ratio of two integers. Nor does the issue of irrationality of numbers have any bearing to their context.

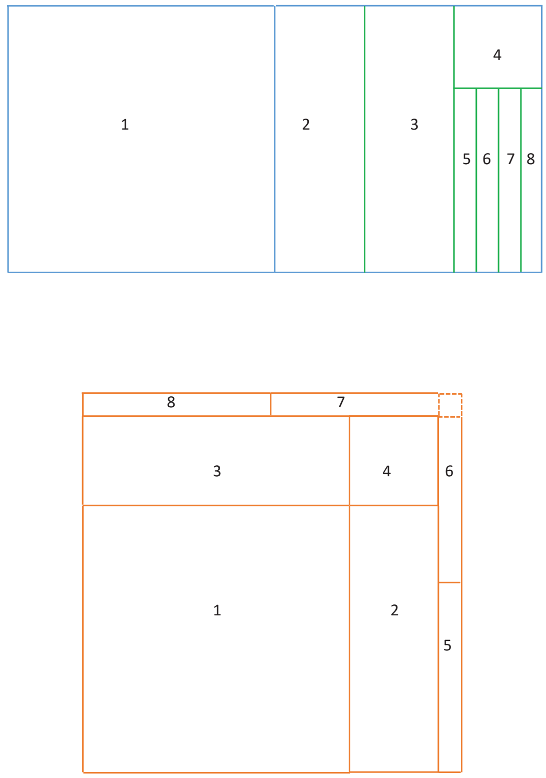

Figure 4. Part of Datta's explanation of Baudhayana's formula for $\sqrt{2}$.

Retain one of the strips on the side of the first square, place another along the perpendicular side in analogy with the geometric procedure. From the remaining strip a square is cut off and is used to fill in the square gap, thus producing a square of side $1 + \frac{1}{3}$ units. We are now left with a strip with sides $\frac{2}{3}$ and $\frac{1}{3}$. Dividing the strip into four equal strips along the longer side, two each are placed along the two sides of the square with side $1 + \frac{1}{3}$. This leads to a square with side $1 + \frac{1}{3} + \frac{1}{3 \times 4}$, whose area is in excess over the desired square, by a square of side $\frac{1}{12}$, as the strips do not cover the tiny square in the corner (on the side of strips 6 and 7 in Figure 4). To adjust for it one subtracts, from the side of the square constructed so far, $\frac{1}{34}$th of the preceding $\frac{1}{12}$th size (this is not shown in our diagram, it being too fine a proportion for the figure); we note that when the tiny square is cut into strips of $\frac{1}{34}$th of the size along one of its sides, by reassembling the pieces along the length we can get two rectangles whose other side is $\frac{17}{12}$, viz., the same length as the side of the last constructed square. This now gives the square with the side as described by Baudhayana. It may be noted that the square is short of the actual value by $\frac{1}{(12 \times 34)^2} \approx 0.000006...$, which may be, and has been, neglected; incidentally, this also explains the matching of the expression with the value of $\sqrt{2}$ upto 5 decimal places.

III.2. The Pythagoras theorem. As noted earlier the Pythagoras theorem was known to the authors of the sulvasutras, the complete statement being found in all the four of them. Here again, one would be curious how they found out that the conclusion holds, and whether they “proved” it geometrically in a suitable sense (though of course they did not have any axiomatic scheme for placing it), as against knowing it simply as a fact about the physical world we live in. There have been a variety of perceptions and speculations in this respect in the literature, but this would not be a place to go into a discussion on it. The theorem is in one way about squares on the sides of the rectangle adding up to the square on the diagonal, and hence relates to the theme we have pursued. Here we content ourselves noting that the statement of the theorem is accessible through constructions akin to the ones we discussed; our formulation here is similar to, but not quite the same, as in [4], pages 115-117.

The argument that we describe is illustrated by Figure 5.[9] Wanting to determine the area of the square ACDE as in the figure one would be inclined to situate it in the larger square BFGH, and observe that the triangular pieces match with those with the corresponding numbers in the other division of the square shown alongside. The comparison then shows that the area of ACDE must coincide with the sum of the areas of the squares numbered 6 and 7, which are indeed the squares of the sides AB and BC respectively.

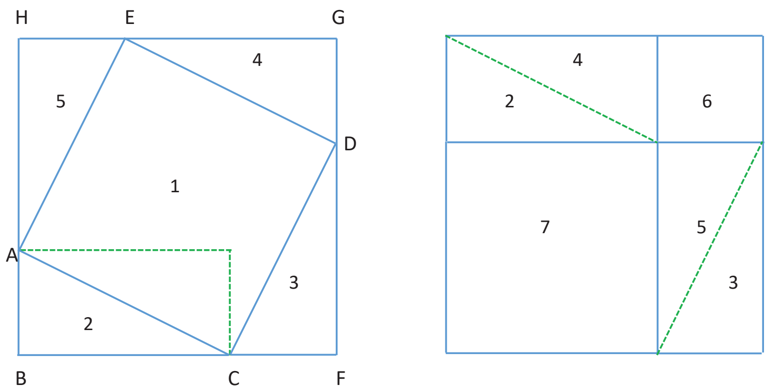

Figure 5. A diagramatic proof of the Pythagoras theorem.

IV. In Place of Conclusion

We have attempted to bring out in this two-part article some of the mathematical essence of the engagement of the authors of the sulvasutras with construction of various planar geometrical shapes with a given area, viz., that of a given square (there are of course other geometric constructions in the sulvasutras, but here we have chosen to focus on the theme as above). The constructions involve a variety of ideas, some at elementary visual level, others in the form of application of the Pythagoras theorem, and yet others certain elements of arithmetic. While in many instances their general scheme of reasoning is clear, in others inevitably the reconstruction of how they arrived at the result is conjectural or speculative, as no proofs are recorded in the sulvasutras. Notwithstanding the lack of certainty or finality the analyses have been instructive in aiding our comprehension of the development of mathematical ideas. It seems especially notable that in the problem of squaring the circle, presumably for want of success in finding a geometric procedure, they appealed to an arithmetical approach, and endeavoured to find a good enough value for $\sqrt{2}$. Unfortunately the full picture with regard to the results is still lacking in clarity, but the process of arriving at it has been instructive. We conclude with the hope that exposure to these ideas would help in enriching geometric understanding of our young readers.

9 It may be borne in mind, however, that the argument presented here is *not* a complete proof of the Pythagoras theorem. In a proof, from a modern perspective, the steps involved need to be based on axioms or known propositions, whereas here various "facts" about planar figures are taken for granted in the course of the argument. In terms of Euclidean geometry, for instance, it needs to be deduced from the basic axioms that the square BFGH involved in the argument can in fact be constructed, and that we can indeed have the partitions into pieces, as posited, that would actually match pairwise, justifying the desired conclusion. These details can indeed be filled in, as an exercise in Euclidean geometry, which we shall not go into here; the issues involved are, however, unlikely to have touched the thought process of the sutrakaras. The above demonstration concerns only the aspect of how the sutrakaras could have convinced themselves of the validity of the theorem, the steps involved being *visually* convincing.

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S.G. DANI was affiliated with the Tata Institute of Fundamental Research (TIFR), Mumbai until retirement. Subsequently, he was with IIT Bombay, and is currently affiliated with the UM-DAE Centre for Excellence in Basic Sciences, Mumbai. He was the President of the *Indian Society for History of Mathematics* for over a decade, and continues to be the Editor of *Ganita Bharati*, the Bulletin of ISHM. He can be contacted at shrigodani@cbs.ac.in.

MEDHA LIMAYE retired as Mathematics Teacher from the Arvind Gandbhir High School, Jogeshwari, Mumbai. She holds a Doctoral degree in Sanskrit, for her thesis on Cultural Aspects of Ancient Indian Mathematics. She has written a book entitled *Treasure of Interesting Numerical Problems*, and several articles on history of mathematics. She is currently serving as Joint Secretary of *Brihanmumbai Ganit Adhyapak Mandal*, Mumbai. She can be contacted at medhalimaye@gmail.com.

Glossary of names and terms

As in text	As in technical literature	In Devanagari script
Apastamba	Āpastamba	आपस्तम्ब
Baudhayana	Baudhāyana	बौधायन
Circle	Maṇḍala	मण्डल
Diagonal	Akṣaṇayā/ Akṣaṇayārajju	अक्ष्णया/अक्ष्णयारज्जु
Flank (longitudinal side)	Pārśvamānī	पार्श्वमानी
Isosceles triangle	Prauga	प्रउग
Katyayana	Kātyāyana	कात्यायन
Manava	Mānava	मानव
Pointed	Aṇimat	अणिमत्
Puruṣa (height of man with uplifted arms)	Puruṣa	पुरुष
Quadrilateral	Caturasra	चतुरस्र
Rectangle	Dīrghacaturasra	दीर्घचतुरस्र
Rhombus	Ubhayataḥ prauga	उभयतः प्रउग
Rope or cord	Rajju, Śulva/Śulba	रज्जु, शुल्व/शुल्ब
Semicircle	Ardhamaṇḍala	अर्धमण्डल
Square (1)	Caturasra	चतुरस्र
Square (2)	Samacaturasra	समचतुरस्र
Stretch	Vistāra	विस्तार
Sulvasutra	Śulva-sūtra/ Śulba-sūtra	शुल्बसूत्र
Sutra (statement in aphoristic style)	Sūtra	सूत्र
Sutrakara (composer of sutras)	Sūtrakāra	सूत्रकार
Transverse (lateral side)	Tiryaṅmānī	तिर्यङ्मानी
Width	Āyāma	आयाम
Yajamana (master of ceremony)	Yajamāna	यजमान
Yajna (fire worship/ritual)	Yajña	यज्ञ