Primary

ClassRoom

Children's Algorithms and the Mathematics behind them

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The first author of this article is a mathematics teacher who observes students' answers, engages with them and is fascinated by them. She approached the second author who works in the field of mathematics education but is not a teacher herself. This article is a collaboration between them.

Due to constraints of time and several other hurdles, many of us do not pause to think about the methods that students use. Why did they think of 'that' particular method? Where did they see it? Will it work for all cases or only special cases?

Most teachers would agree that many of their students have alternative ways of solving problems. Sometimes, as teachers we are not even sure why these methods work but they seem to work.

These alternative methods and discussions on them make the classroom space richer and a more democratic one, as a student who comes up with an interesting method of his or her own may not, usually, be a mathematics enthusiast. She could be the one who hardly participated or was otherwise a mathematical introvert. Giving spaces to students' own methods also offers opportunities to explore new ideas with the reins of discussions in the students' hands.

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Both authors find studying students' own ways of solving problems very exciting. That is the main reason behind this collaboration. In this article, we would be talking about different methods of solving problems that students from the first author's class gave her.

Let us peep into one of the classrooms:

The teacher asked the students to solve 45 – 19. She knew that the students were comfortable in subtracting numbers when the units digit of the subtrahend was smaller than the units digit of the minuend, that is when no borrowing was needed. This particular question was given to see how students deal with problems where the 'borrowing' technique is typically used.

Sai was seen solving this method using 'number completion' method.

$$\begin{array}{r} 20 \\ 45 \\ -\ 19\nearrow \\ \hline 26 \end{array}$$

Teacher: Sai, why have you put an arrow and written 20 here?

Sai: The closest 'tens number (दशक संख्या)' (multiple of 10) to 19 is 20. I found subtracting 20 from 40 more convenient.

Teacher: Where did you see this way of calculating?

Sai: Tai[1], The bus conductor also calculates like this.

Teacher: I would like to understand your method.

Sai: The closest 'tens number (दशक संख्या)' (multiple of 10) to 19 is 20. So, I subtracted 20 from 40 and got 20. Then added 5 ones to 20.

Teacher: But then the answer is 25. But you have written 26 instead.

Sai: The closest 'tens number (दशक संख्या)' (multiple of 10) to 19 is 20, which is 1 more than 19. While subtracting, I subtracted 20 from 45 and I got 25. So then, I added that 1 to 25 and got 26.

In the given transcript, one can see Sai using a 'number completion' method wherein the number closer to a multiple of 10 (here, 19) was completed to its nearest ten, followed by subtraction and then compensating for 'completion'. This is not done traditionally in a usual class. The teacher is seen asking questions to understand the method from Sai. This approach would not only encourage Sai and her classmates to use their own methods to solve problems but would also help Sai in articulating her method better. Sai had observed her bus conductor using this strategy and she applied it to her class problem.

Let us look at some more examples of how Sai and her classmates solved problems by their own methods in an atmosphere which not only allows but also encourages them to use their own algorithms.

Teacher asked Sai another question:

$$\begin{array}{r} 47 \\ -\ 29 \\ \hline \end{array}$$

Sai clearly ignored the vertical arrangement of the numbers which was written to facilitate the *'borrrowing'* method and solved the problem in the following way.

"I subtracted 20 from 40, then I subtracted 9 from the difference (20), 11 remains.

Then I added 7 to the 11 so that the answer is 18."

A few things that one notices about Sai's method are:

Sai is looking at the numbers 47 and 29 as a whole not as individual digits 4, 7 and 2, 9. For her, she decomposes the numbers 47 and 29 as 40 + 7 and 20 + 9 respectively.

[1] Tai means elder sister in Marathi but here the students are using it to address their teacher.

Let us solve another example using Sai's number decomposition method.

$$\begin{array}{r} 327 \\ -\ 258 \\ \hline \end{array}$$

We can imagine that Sai would solve this in the following way.

$$\begin{array}{r} 327 \\ -\ 258 \\ \hline 69 \end{array}$$

She would subtract 250 from 300, followed by subtracting 8 from the difference (50), 42 remains.

Finally, Sai would add 27 to the 42 to get the answer 69.

Or, she may do the following:

Subtracting 200 from 300, followed by subtracting 50 from the difference (100), 50 remains. And, then subtracting 8 from 50 to get 42. Finally, adding 27 to get 69.

Let us understand the decomposition strategy used by Sai.

Let us think of two 2-digit numbers, $N = a_1 \times 10 + a_0$ and $M = b_1 \times 10 + b_0$

(Note that all *a*'s and *b*'s stated here are digits, that is, each of them is a numeral from the set 0 to 9.)

First subtract the digits at the largest place. Here, since the largest place holders are at 'tens' place, so perform $10a_1 - 10b_1 = 10\ (a_1 - b_1)$

Then, subtract the units digit of the subtrahend from the difference obtained, $10(a_1 - b_1) - b_0$. This is fairly simple because of the multiple of 10 present in the subtraction.

Finally, add the units digit of the minuend $(10(a_1 - b_1) - b_0 + a_0 = N - M)$

By using her own algorithm, Sai has completely circumvented the problem of borrowing, a process which can be very troublesome for some children.

We could see that Sai used her indigenous algorithm to subtract numbers that involved borrowing. She found this algorithm more comfortable than the traditional borrowing method. Her algorithm could be used for any subtraction problem involving whole numbers. One may ask questions like "Can this be done for all numbers?", "Would the strategy change from number to number?", "Can this be an alternative algorithm for subtraction of numbers?"

Over the years, many people working in mathematics education have talked about children's methods of solving mathematical problems. Whenever questions such as, "Is it applicable for all numbers?" are asked, these are countered by saying, "Why should it work for all numbers for it to be interesting?" There is some truth in what they say. But if a child sees a pattern and hence develops a method to solve a particular type of problem, it would be interesting to look at the underlying mathematical principles that she used. If one is convinced by the underlying mathematical principles that were adopted, one may decide if the developed algorithm may be used only in particular situations or these can be generalized.

Though Sai's algorithm is fairly simple, we will see that sometimes children may come up with rather complicated algorithms. When Sai's classmates and schoolmates were asked the same problems, they came up with more interesting methods. Let us look at some of them:

When Poornima was asked to solve 47 – 29, she first subtracted 30 from 40 saying that it was the closest tens-number (दशक-संख्या), and she got 10. She then added 7 to the 10 and got 17. Then, she reminded her teacher that she had subtracted one extra from 40 (instead of 29 she had subtracted 30) and so, 1 is to be added to 17, thus making the answer 18.

Another student, Kajal, used a different method and following is the conversation between Kajal and her teacher.

Teacher: Kajal, what is your answer to 47 – 29?

Kajal: 18, because 4 tens means 40. From these, I subtract 2 tens or 20. So I have 20 with me. While subtracting 9 from 7, I have 2 less. These two I subtract from 20, so that the answer is 18.

Poornima and Kajal are both Sai's classmates. Looking at the various methods children use to solve these problems one wonders, "Who taught them these alternative methods?"

When Sai was asked from where she picked up her method, she said that she sometimes accompanies her mother who serves as home help in many houses. She has often seen her mother doing *hisaab* (calculations) that way. Similarly, Kajal had also seen her mother doing calculations when she sells corn. Both children had adopted the day-to-day techniques learnt by accompanying their elders.

For both of them, their lives outside school have helped them develop their mathematics within the classroom. Both of them seem very comfortable in using non-traditional algorithms adopting them in formal school problems. Although they were familiar with the techniques informally, they developed them into formal techniques by using the terminology used in school, like tens and units (ones) which is never used in out-of-school calculations.

All of us have seen such algorithms emerging in our classrooms. One such example was also observed by the second author with another student, Aman. Aman was very comfortable with multiplication facts of 12, possibly because he used to help his brother deliver eggs to shops and hotels. He solved problems like 13 × 11 as follows:

13 × 11:

13 × 12 = 156

13 × 11 = 156 – 13 = 143.

His daily activities helped Aman remember the multiplication facts of 12 but he figured out the relationship between 13 × 12 and 13 × 11 himself.

Children who, as part of their routine, take various responsibilities in their family are more likely to design their own approaches to learning. These approaches draw on what they have learnt while fulfilling their work or other responsibilities. These instances indicate that students notice mathematical patterns outside the classroom. It is another matter whether they get a chance to share what they notice. Some of the examples above, also illustrate how their noticing is coupled with formal terminology of mathematics, either to make it as an algorithm for a classroom or to take advantage of the compactness of the terminology.

Another student from Sai's school was asked

$$\begin{array}{r} 53740 \\ -\,38999 \\ \hline \end{array}$$

This was the algorithm he used:

5 TT (ten thousands) + 3 Th (thousands) + 7 H (hundreds) + 4 T (tens) + 0 O(ones)

$$\begin{array}{rrrrr} 4\text{ TT} + & 12\text{ Th} + & 16\text{ H} + & 13\text{ T} + & 10\text{ O} \\ -\,3\text{ TT} + & 8\text{ Th} + & 9\text{ H} + & 9\text{ T} + & 9\text{ O} \end{array}$$

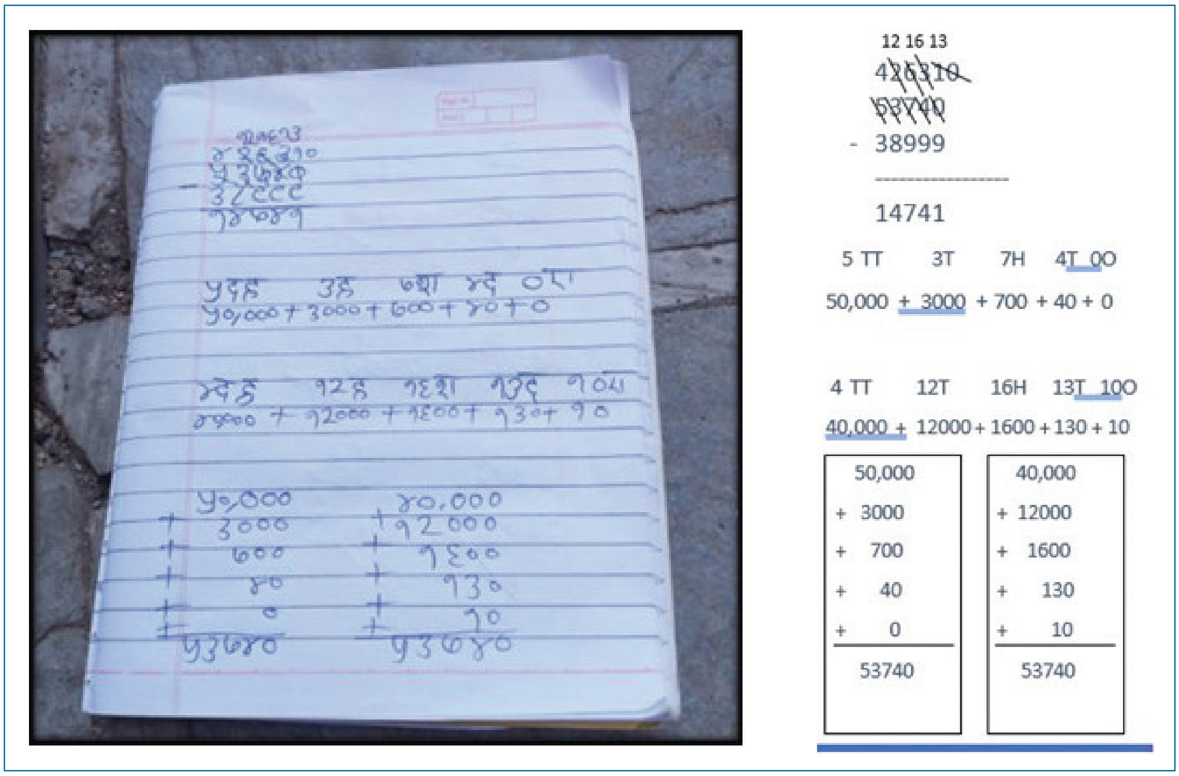
= 1 TT + 4 Th + 7H + 4 T + 1 O

= 14741

When asked how he was sure of his answer, he went on to demonstrate his method:

5 TT (ten thousands) + 3 Th (thousands) + 7 H (hundreds) + 4 T (tens) + 0 O(ones) is equal to 4 TT + 12 Th + 16 H + 13 T +10 O by actually adding each of them.

Look at the image attached.



Let us solve another problem using this student's method.

$$\begin{array}{r} {\scriptstyle 4\ 16\ 3\ 13} \\ \cancel{5}\,\cancel{6}\,\cancel{4}\,\cancel{3}\,\cancel{2} \\ -\ 4\ 8\ 3\ 5\ 1 \\ \hline 0\ 8\ 0\ 8\ 1 \end{array}$$

Let us try to see the *'mathematics'* behind his answer.

Let us think of a five-digit number, N = $a_4\, a_3\, a_2\, a_1\, a_0$ so the expanded form of this number is $a_4 \times 10000 + a_3 \times 1000 + a_2 \times 100 + a_1 \text{x } 10 + a_0$.

Now we can rewrite this as,

$(a_4 - 1) \times 10000 + (9 + a_3\,) \times 1000 + (9 + a_2\,) \times 100 + (9 + a_1) \times 10 + (10 + a_0\,)$.

So we have to find the final form of the answer to: $a_4\, a_3\, a_2\, a_1\, a_0 - b_4\, b_3\, b_2\, b_1\, b_0$

(Note that if $a_4 = 1$, $a_3 = 2$, $a_2 = 3$, $a_1 = 4$, $a_0 = 5$, then $a_4\, a_3\, a_2\, a_1\, a_0 = 12345$ and not the product of 1, 2, 3, 4 and 5.)

Case 1: $a_4 > b_4$

Then,

$a_4\, a_3\, a_2\, a_1\, a_0 - b_4\, b_3\, b_2\, b_1\, b_0$

$= (a_4 - 1 - b_4) \times 10000 + (9 + a_3 - b_3) \times 1000 + (9 + a_2 - b_2) \times 100 + (9 + a_1 - b_1) \times 10 + (10 + a_0 - b_0)$

You can check that this when added to $b_4\, b_3\, b_2\, b_1\, b_0$ gives you $a_4\, a_3\, a_2\, a_1\, a_0$

Case 2: $a_4 = b_4$

Then the answer will be: $(a_3 - 1 - b_3) \times 1000 + (9 + a_2 - b_2) \times 100 + (9 + a_1 - b_1) \times 10 + (10 + a_0 - b_0)$

Case 3: $a_3 < b_3$

This will lead to an answer in integers which is beyond the scope of this article.

These algorithms work on all subtraction sums, even those with more than 5 digits! You have to try writing the same algorithm for numbers with more digits. Additionally, we have here algorithms where you can do subtraction from left to right, something which is not followed in ritualistic school-taught procedures.

Looking at these different methods used by the students, one can't help but wonder about their school and their society. All these students were from a government school in the district of Ahmednagar, in Maharashtra. The school is run by the local government, Nagar Palika.

Most of these children belong to Nomadic Tribe communities. Many of them are also first -generation school goers. Most parents leave home early in the morning and engage in small businesses such as selling corn. These students come from extremely marginalized communities.

If given a chance, where rote learning is not given a premium and independent thinking not penalized, it is likely that children come up with very different and novel methods of solving problems, that are more comprehensible to them and not a mere mechanical application of formulae or the use of algorithms which are not transparent such as the division algorithm.

Most students in these classes were seen not being scared of mathematics as their engagement in mathematics was always through their own contexts. They were seen attaching different meanings to the same operation. Often, unfamiliar problems asked in the classroom, with minimal instructions motivate students to find newer ways, apply their prior understanding, use methods which are familiar to them from their outside the classroom experiences. Their success in solving these unfamiliar problems increases their confidence in doing mathematics. They also tend to be more open to others' methods and they try to understand the solutions given by their classmates. The reason the students are so articulate in expressing their way of solving might be because giving reasons for their own methods is encouraged in their class and is an important part of their classroom culture.

It is very heartening to see more and more teachers encouraging children to think and devise methods to solve problems differently and innovatively without being bogged down by rote learning or mechanically following the formulae or algorithms. Encouraging students to devise their own methods is a powerful approach in bringing out a creative, thinking and rational student body.

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