Tom and Jerry Play with Fractions

PRITHWIJIT DE

In the current school term in Cat and Mouse Academy, Tom, the cat, is teaching algebra to Jerry, the mouse. Having covered the basics Tom is now in a mischievous mood and is challenging Jerry with brainteasers. Instead of revealing the challenge at once he builds it up gradually in his characteristic style.

So he writes on the board:

$$\frac{1}{4}=\frac{16}{64}=\frac{166}{664}=\cdots=\frac{1\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}4}=\cdots,$$

turns to Jerry and asks him if the relationship is true for all integers $n \geq 0$.

Little Jerry quickly checks that

$$\frac{1}{4}=\frac{16}{64}=\frac{166}{664}=\frac{1666}{6664}$$

and wonders how to tackle the general term. After some time, he figures out that

$$\underbrace{666\ldots6}_{n}4=60\left(\frac{10^n-1}{9}\right)+4=4\left(\frac{15\cdot10^n-6}{9}\right)$$

and

$$1\underbrace{666\ldots6}_{n}=10^n+6\left(\frac{10^n-1}{9}\right)=\frac{15\cdot10^n-6}{9}$$

to observe to his delight that indeed

$$\frac{1}{4}=\frac{1\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}4}.$$

Keywords: Decimals, recurrence, fractions, patterns

Moving on. Impressed by Jerry's response, Tom moves a step ahead. He asks Jerry to find out if there are other proper fractions with the same property. To help Jerry understand the problem, Tom spells it out in clear terms as stated below.

Challenge problem. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find all $a, b \in S$ with $a < b$ such that given a and b, there exists $c \in S$ (with $c \neq a,\ c \neq b$) for which

$$\frac{a}{b} = \frac{a\overbrace{cc\ldots c}^{n}}{\underbrace{cc\ldots c}_{n}b}$$

for all integers $n \geq 0$.

With his task cut out, Jerry summons up courage and begins his investigation. He immediately observes that he had already found $(a, b, c) = (1, 4, 6)$ to be a solution and decides to mimic the steps that he followed to find that solution. After a brief bout with algebra, he obtains

$$\frac{a}{b} = \frac{(9a + c)10^n - c}{c.10^{n+1} + 9b - 10c}$$

which leads to

$$9a(c - b) = c(b - a).$$

He observes that 9 divides $c(b - a)$ and $(b - a) \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Now Jerry makes a clever move as he often does to heckle Tom. He considers two cases:

(I) 3 does not divide $b - a$;

(II) 3 divides $b - a$.

In case (I) he readily obtains $c = 9$ and

$$b = \frac{10a}{a + 1} = 10 - \frac{10}{a + 1}$$

to conclude that $a + 1$ must divide 10. So $a \in \{1, 4\}$ and $(a, b, c) = (1, 5, 9)$ or $(4, 8, 9)$.

In case (II) he observes that $b - a \in \{3, 6\}$ and decides to treat the two sub-cases separately. If $b - a = 3$ then he obtains

$$c = \frac{3ab}{3a - 1} = \frac{3a(a + 3)}{3a - 1} = a + 3 + \frac{a + 3}{3a - 1}$$

and asserts that for c to be an integer $3a - 1$ must divide $a + 3$ and since both are positive integers, $3a - 1 \leq a + 3$ or $a \leq 2$. He quickly disposes of the two possibilities $a = 1$ and $a = 2$ to obtain $(a, b, c) = (1, 4, 6)$ or $(2, 5, 6)$.

If $b - a = 6$, he finds

$$c = \frac{3a(a + 6)}{3a - 2}$$

and uses the fact $c \leq 9$ to obtain

$$9 - \frac{3a(a + 6)}{3a - 2} \geq 0,$$
$$\therefore\ a^2 - 3a + 6 \leq 0 \quad \text{(after simplification)}.$$

But

$$a^2 - 3a + 6 = \left(a - \frac{3}{2}\right)^2 + \frac{15}{4} \leq 0,$$

an absurd result. So he rules out this possibility.

Having exhausted all possibilities he jumps in joy and claims that the only possible fractions are

$$\frac{1}{5} = \frac{1\overbrace{999\ldots9}^{n}}{\underbrace{999\ldots9}_{n}5}, \qquad \frac{1}{4} = \frac{1\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}4},$$

$$\frac{2}{5} = \frac{2\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}5}, \qquad \frac{4}{8} = \frac{4\overbrace{999\ldots9}^{n}}{\underbrace{999\ldots9}_{n}8}.$$

The show is not over. Highly encouraged by this success Jerry asks the following question.

Question. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find all $a, b \in S$ with $a < b$ such that given a and b, there exists $c \in S$ (with $c \neq a,\ c \neq b$) for which

$$\frac{a}{b} = \frac{\overbrace{ccc\ldots c}^{n}a}{b\underbrace{ccc\ldots c}_{n}}$$

for all integers $n \geq 0$.

Tom is still searching for the answer. Can you help Tom?

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.