Secondary

Problem Corner

A Property of the Centroid of a Triangle

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In this note, we establish an unexpected property of the centroid of a triangle.

Given any triangle ABC, let P be an arbitrary point lying within the triangle. Drop perpendiculars PD, PE, PF to the sides BC, CA, AB respectively. We ask: For which point P does the product $PD \cdot PE \cdot PF$ take its largest possible value?

We shall show that for any triangle, *$PD \cdot PE \cdot PF$ takes its maximum value when P lies at the centroid of the triangle.*

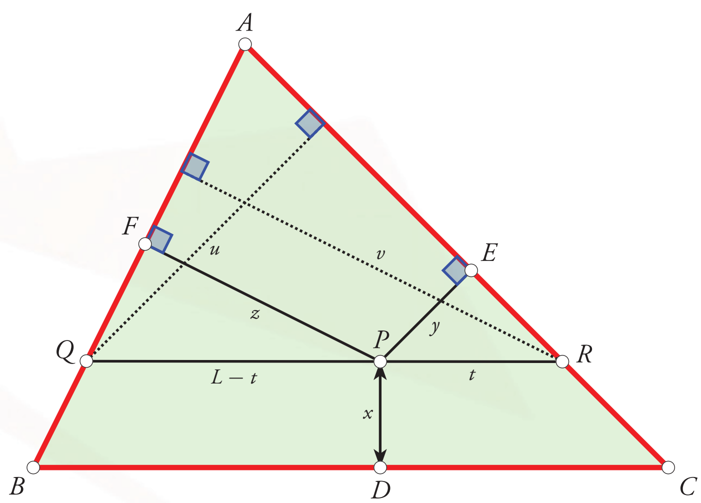

Figure 1.

Keywords: Centroid, perpendicular, product, similar triangles, quadratic, maximum

For any point P within the triangle, let x, y, z be the distances from P to BC, CA, AB respectively ($x = PD$, $y = PE$, $z = PF$). To start with, let x be kept constant. The set of points P such that x has any given fixed value is a segment QR parallel to side BC and at distance x from it (see Figure 1). Let u and v be the perpendicular distances from Q to AC and R to AB, respectively. Let L be the length of QR. Let $PR = t$; then $QP = L - t$.

Using similarity we have:

$$\frac{y}{u} = \frac{PR}{QR}, \quad \therefore\ y = \frac{tu}{L},$$
$$\frac{z}{v} = \frac{QP}{QR}, \quad \therefore\ z = \frac{(L-t)v}{L}.$$

For any given fixed value of x, both u and v are constants, as is L. Hence:

$$yz = t(L-t) \times \text{some constant which depends only on } x.$$

The variable component on the right side is the quadratic expression $t(L - t)$, which achieves its maximum value when $t = L/2$, i.e., when P lies at the midpoint of QR. (Recall that if the sum of two non-negative numbers is a positive constant, then their product takes its largest value when the two numbers are equal.)

So, for each value of x, the optimal location of P is the midpoint of QR. That is, for each value of x, the optimal location of P lies on the median of the triangle through vertex A.

By symmetry, the optimal location of P must also lie on the medians through vertices B and C. This implies that the optimal location is the centroid of the triangle.

Since the perpendicular distance of the centroid from each side is $1/3$ of the corresponding altitude, it follows that the largest possible value of the product of perpendicular distances from the sides is equal to $1/27$ of the product of the three altitudes. □

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