Secondary

Problem Corner

The Bevan Point and Associated Points and Circles

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The following note combines aspects of mathematics (geometry), history, heuristics (problem-solving), dynamic geometry software and pedagogy. Although it may go a bit beyond the curriculum at school, the mathematics and arguments stay elementary. In the second part it should be a vivid demonstration that simply trying things and seeing what happens – i.e., fostering the process of 'doing mathematics' – can lead to nice results that can also be useful for teaching situations.

In 1804 the British engineer Benjamin Bevan posed a problem [1] of Euclidean geometry (it was solved in the same year by John Butterworth):

Problem. Given a triangle $\triangle ABC$ with incentre I, circumcentre O, and excentres I_a, I_b, I_c. Let V be the circumcentre of the excentral triangle $\triangle I_aI_bI_c$ (see Figure 1). Prove the following:

(1) V is the reflection of I in O;
(2) The circumradius of $\triangle I_aI_bI_c$ is twice the circumradius of $\triangle ABC$.

General remarks. Owing to the history of the problem, the circumcircle of the excentral triangle $\triangle I_aI_bI_c$ is called the *Bevan circle* and its centre the *Bevan point* (point X(40) in Clark Kimberling's *Encyclopedia of Triangle Centers* or 'ETC'). In some approaches the nine-point-circle is involved (especially when other properties also need to be proved), but if one restricts to (1) and (2), a weaker result suffices, which is accessible also to students, e.g., in a problem-solving class. Let us have a closer look at the above. If we can prove that the circumcircle of $\triangle ABC$ bisects the segments joining the incentre I to the excentres (points of bisection being D, E, F, see Figure 2; let us call this Lemma 1), then we are done, because then we can conclude: Under an enlargement (also known as a 'homothety') with centre I and scale factor 2, the circumcircle of $\triangle ABC$ is mapped to the circumcircle of $\triangle I_aI_bI_c$, and this proves both (1) and (2).

Keywords: Bevan circle, Bevan point, incentre, circumcentre, excentre, circumradius, dynamic geometry, enlargement, homothety

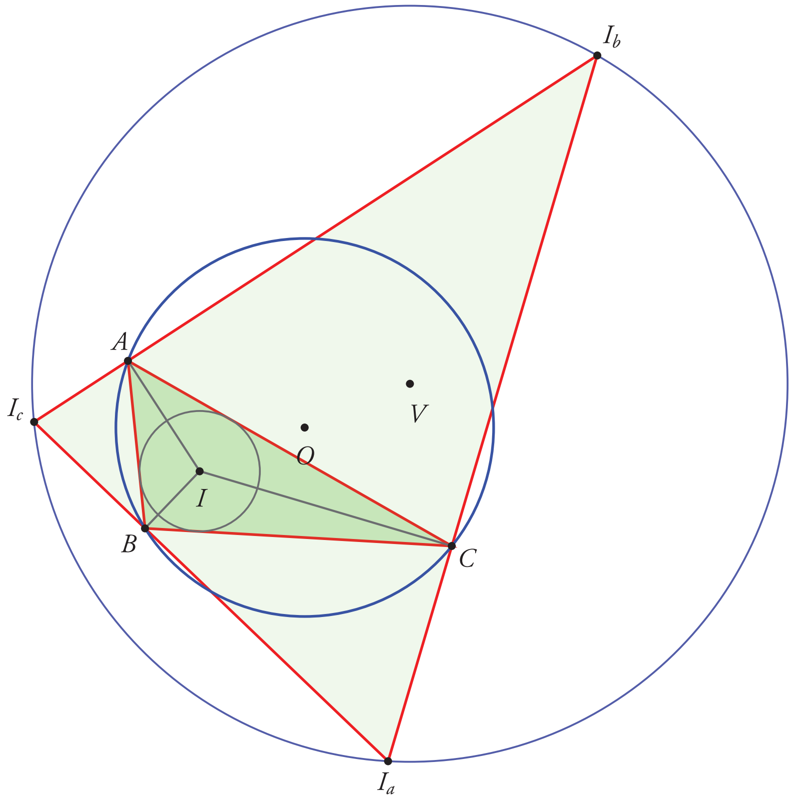

Figure 1. The Bevan circle, centre V

Proof of Lemma 1. We consider the connector II_a (see Figure 2; the others work the same way). Let D be the intersection point of the line segment II_a and the arc BC of the circumcircle of $\triangle ABC$; this must be the midpoint of arc BC, because AD is the bisector of angle BAC; this also implies that $DB = DC$. Quadrilateral BI_aCI is cyclic (note the right angles at B and C), and II_a is a diameter of its circumcircle, so the centre of the circumcircle lies at the midpoint of II_a. Angle computations show that $DB = DI = DC$ (for $\measuredangle DBI = \measuredangle DBC + \measuredangle CBI = (A+B)/2$, and $\measuredangle DIB = (A+B)/2$ too); so D is the centre of the circumcircle. Therefore D bisects II_a (see [3], p. 185 and p. 192).

Many references can be cited for the results quoted above; the facts are quite well known.

Exploration. Now we come to the second part of the article, which is exploratory in nature. We interchange one of the excentres with the incentre in the process described above and see what happens (thereby fostering the process of 'doing mathematics'). For instance, take I instead of I_c and find the circumcentre of $\triangle I_aI_bI$ and label it V_c. Do the same with I instead of I_a (giving V_a), and I instead of I_b (giving V_b), respectively. What we get is a triangle $V_aV_bV_c$ that is nothing but triangle $I_aI_bI_c$ reflected in point O (see Figure 3). Initially we could not find any reference concerning this fact, but then colleagues pointed us to [5] (maybe the above phenomenon is described there for the first time? This approach is completely different from the one presented here) and also to [6] (p. 110, Ex. 10) ([5] is cited here too). Overall, we have the impression that this phenomenon is not so well known, In addition, this topic may be used in a geometry class, thus we wrote this short note.

Let us initially consider the first case (I instead of I_c, giving V_c). The new circle passing through I_a, I_b, I has properties similar to the Bevan circle above:

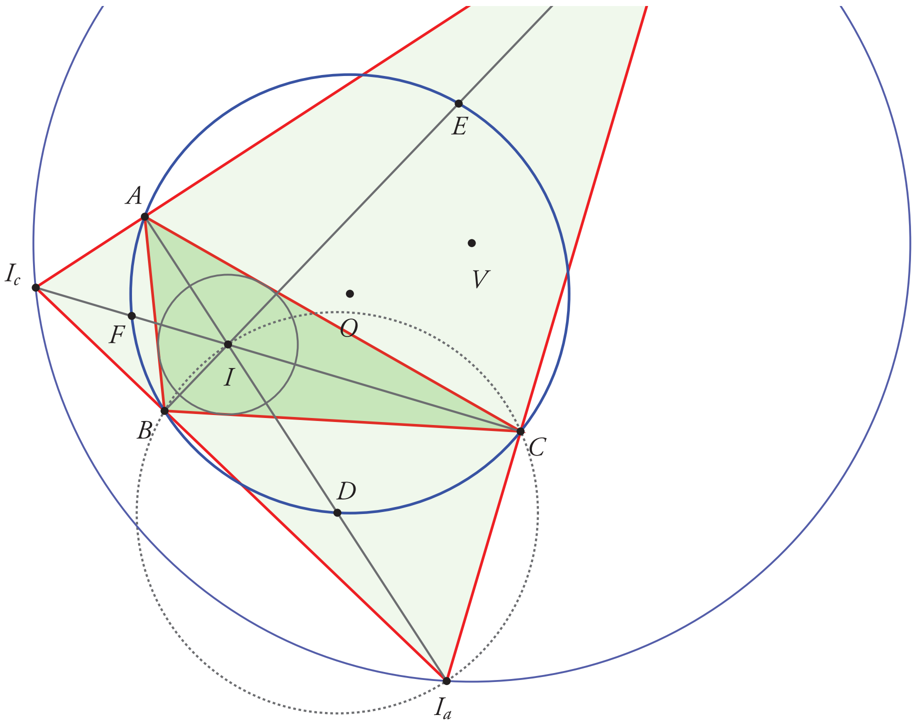

Figure 2. Why is segment II_a bisected by D?

(a) Its radius is twice the circumradius of $\triangle ABC$;

(b) Its centre V_c is the reflection of I_c in O.

The two other cases behave analogously. Altogether, we can think of five equally large circles:

(1) Bevan circle through I_a, I_b, I_c (centre: V)

(2) Circle through I_a, I_b, I (centre: V_c)

(3) Circle through I_a, I, I_c (centre: V_b)

(4) Circle through I, I_b, I_c (centre: V_a)

(5) Circle through V_a, V_b, V_c (centre: I)

In some sense, we could call all these circles 'Bevan circles' and all the points V_a, V_b, V_c 'Bevan points.'

In order to prove that $V_aV_bV_c$ is the triangle $I_aI_bI_c$ reflected in the point O, it suffices to show a) and b) from above (the proofs for the other points V_a, V_b work the same way). As before we are done if we can prove that the circumcircle of $\triangle ABC$ bisects the segments joining I_c to I_a, I_b, I (see Figure 4). We proved already that D bisects the line segment II_c. As above we see (note the right angles at C and B) that I_bI_cBC is a cyclic quadrilateral. Let G be the intersection point (other than A) of line segment I_bI_c and arc CB (in other words, the midpoint of the arc CB). Then we know that $GC = GB$ and because of the cyclic quadrilateral I_bI_cBC (for which I_bI_c is a diameter) we know that $GI_b = GC = GI_c$, and this means that G bisects I_bI_c (and similarly, H bisects I_aI_c).

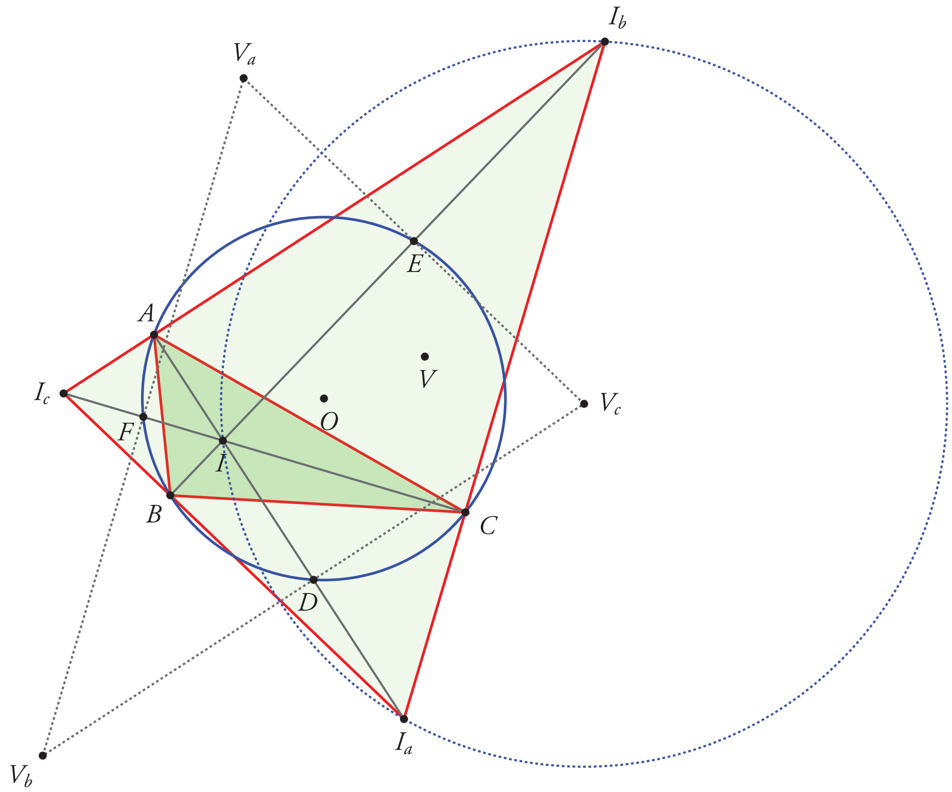

Figure 3. The three points V_a, V_b, V_c and the circumcircle of $\triangle I_a I_b I$

Teachable points. Possible teaching situations concerning this topic could be the following.

(I) Problems (1) and (2) posed at the start can be an occasion for autonomous problem solving by gifted students. The teacher may need to give several hints (dealing with excentres is not self-evident for students in most countries):
  (a) Why do segments AI_a, BI_b, CI_c all pass through I (Figure 2)?
  (b) Why does it suffice to show that the circumcircle of $\triangle ABC$ bisects the segments connecting the incentre I and the excentres?
  (c) How can we be sure that there are right angles at points B and C (Figure 2)?
  (d) Why is D the centre of the circumcircle of cyclic quadrilateral IBI_aC (Figure 2)?

  By thus dividing the problem into smaller pieces, problem solving and proving may become more manageable for the students, enabling them to find correct arguments.

(II) (1) and (2) can also be demonstrated by a teacher, after which (a) and (b) can serve as exercises for students (not just gifted students) for practising techniques they have learnt.

In both cases dynamic geometry software can be used for setting up conjectures and seeing all the results (which gives a strong hint that the claims are true). Of course, the proof must be provided by the learners; here the computer cannot help. In teaching, to see *why* something is true (explanation function of proof) is even more important than to see that something is true (verification function of proof, [2]).

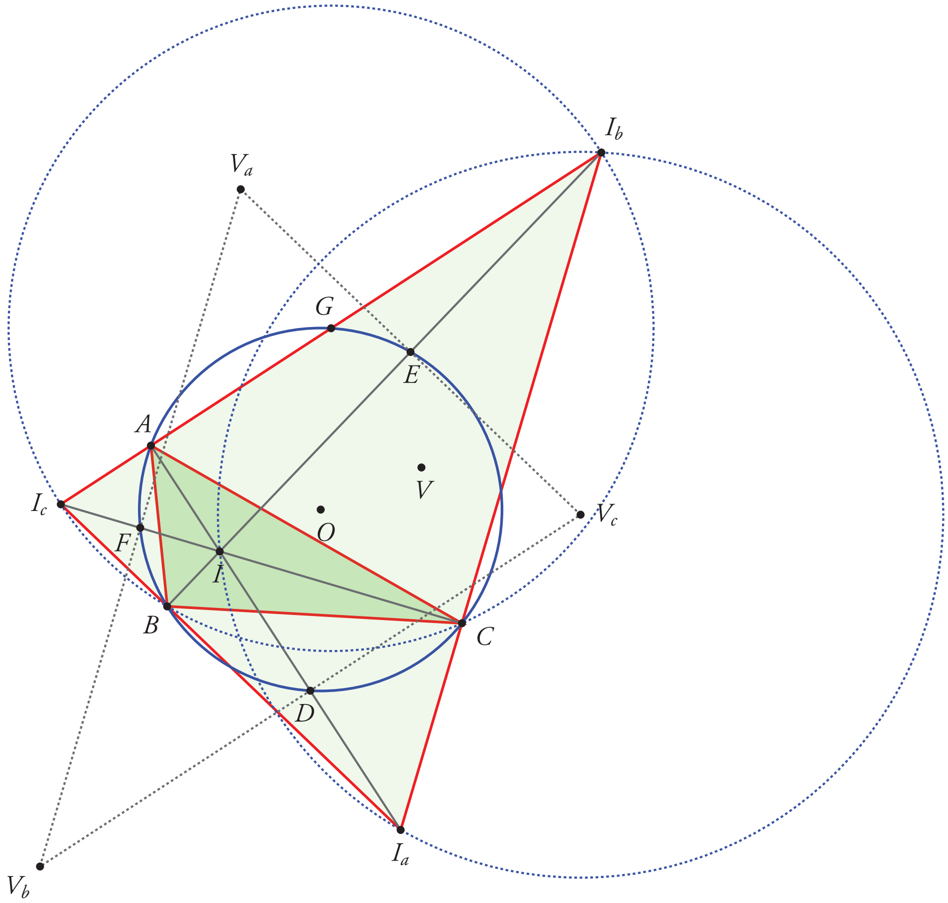

Figure 4. Why is I_bI_c bisected by G?

References

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HANS HUMENBERGER has been a professor of mathematics education at the University of Vienna since 2005. In this position, he is responsible for the education of students who want to become teachers at secondary schools or high schools (grades 5-12, 11-18-year-old students). His primary interests include teaching mathematics as a process, teaching mathematical modeling, problem solving, geometry, and elementary mathematics. He may be contacted at hans.humenberger@univie.ac.at.

Addendum: The nine-point circle of a triangle

Reference has been made in the above article to the nine-point circle of a triangle. Not all readers may be familiar with this notion, so the editors have included this addendum.

References to the nine-point circle are easily found online. See for example:

- "Exploring the Nine-point Circle: Conjecture making and proof using Dynamic Geometry Software" by Jonaki Ghosh in the March 2021 issue of *At Right Angles*
- https://en.wikipedia.org/wiki/Nine-point_circle
- https://artofproblemsolving.com/wiki/index.php/Nine_point_circle

Let ABC be any triangle. Let D, E, F be the midpoints of sides BC, CA, AB, respectively. (See Figure 5.) Let the altitudes of the triangle be AP, BQ, CR, with P, Q, R lying on BC, CA, AB, respectively. Let H be the orthocentre of the triangle (i.e., the common point of the three altitudes). Let U, V, W be the midpoints of HA, HB, HC, respectively. Let O be the circumcentre of $\triangle ABC$. Then it turns out that the nine points $D, E, F, P, Q, R, U, V, W$ lie on a circle. This circle is known as the **nine-point circle** of $\triangle ABC$. Let N be its centre. It turns out that N lies at the midpoint of segment OH. (The centroid G of $\triangle ABC$ also lies on segment OH.)

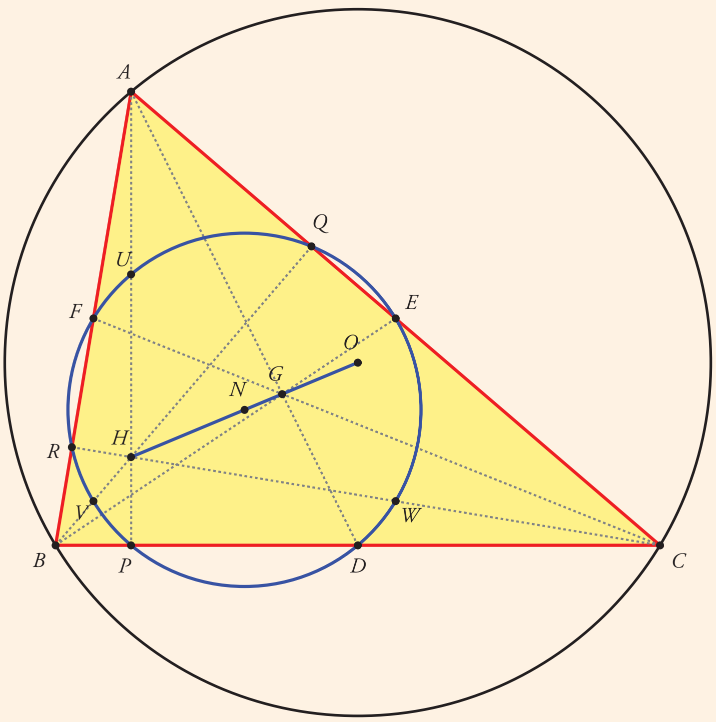

Figure 5