Infinite Kepler Triangles

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In this article, we shall explain what a Kepler triangle is and describe an infinite nested array of such triangles.

Introduction

The Fibonacci numbers form one of the most intriguing sequences of natural numbers. The sequence goes $1, 1, 2, 3, 5, 8, \ldots$. Here the n-th Fibonacci number F_n (for $n \geq 3$) can be expressed as the sum of the previous two Fibonacci numbers, i.e.,

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).$$

It is well-known that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi,$$

where $\varphi = (1+\sqrt{5})/2 \approx 1.618$ is the **golden ratio**. Note that φ is the positive root of the quadratic equation $x^2 - x - 1 = 0$. So the golden ratio satisfies the relation $\varphi^2 = \varphi + 1$. A consequence of this basic relation is the following:

$$\varphi^{n+1} = F_{n+1}\varphi + F_n, \quad n \geq 1.$$

The Kepler triangle

An interesting occurrence of the golden ratio in Euclidean geometry is in the Kepler triangle. A Kepler triangle is a right-angled triangle whose sides are in geometric progression. If the common ratio of this geometric progression is $\sqrt{x}$, then the sides are in the ratio $1 : \sqrt{x} : x$, so we have $1 + x = x^2$, which shows that $x = \varphi$. So the sides of the Kepler triangle are in the ratio

$$1 : \sqrt{\varphi} : \varphi \approx 1 : 1.272 : 1.618.$$

Keywords: Kepler triangle, Fibonacci numbers, Golden ratio

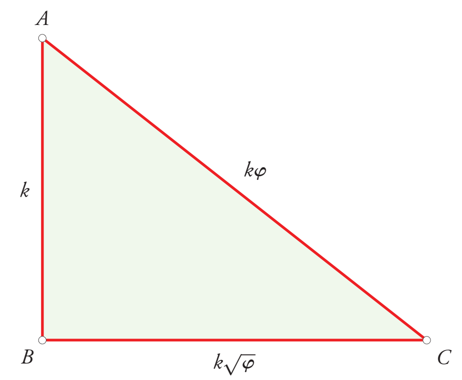

Figure 1. Kepler Triangle

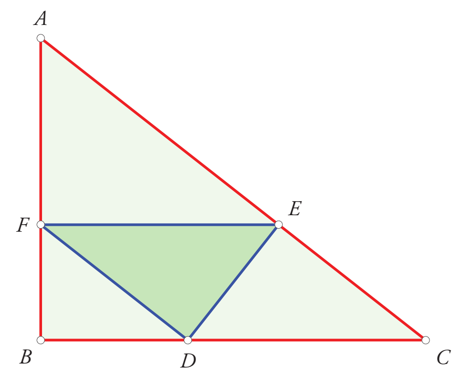

Figure 2. Constructing $\triangle DEF$ from $\triangle ABC$

See Figure 1. Note that a triangle with sides $k,\ k\sqrt{\varphi},\ k\varphi\ (k>0)$ is also a Kepler triangle since

$$(k\varphi)^2 = (k\sqrt{\varphi})^2 + k^2.$$

Clearly, a triangle that is similar to a Kepler triangle is itself a Kepler triangle.

The problem

Let us now do something interesting. Let $\triangle ABC$ be a Kepler triangle with hypotenuse AC; let its sides AB, BC, CA be $k,\ k\sqrt{\varphi},\ k\varphi$. Let the sides be divided by points F, D, E, respectively, such that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{BF}{FA} = \frac{1}{\varphi}. \tag{1}$$

That is, each side is divided in the 'golden section' (see Appendix for details). See Figure 2. The question now is: What can be said about the four triangles thus created: $\triangle DEF$, $\triangle AFE$, $\triangle BDF$, $\triangle CED$? In general, this process can be continued to get more nested triangles; what can be said about them?

Let $BF = x$, $AF = \varphi x$; $BD = y$, $DC = \varphi y$; $CE = z$, $AE = \varphi z$ for some x, y, z. Then:

$$x(\varphi+1) = k, \quad \therefore\ x = \frac{k}{\varphi+1}. \tag{2}$$

Similarly:

$$y(\varphi+1) = k\sqrt{\varphi}, \quad \therefore\ y = \frac{k\sqrt{\varphi}}{\varphi+1}, \tag{3}$$

and

$$z(\varphi+1) = k\varphi \quad \therefore\ z = \frac{k\varphi}{\varphi+1}. \tag{4}$$

Observe that

$$x : y : z = 1 : \sqrt{\varphi} : \varphi. \tag{5}$$

Now note that FE is parallel to BC (because $AF/FB = AE/EC$); hence $\triangle AFE$ is similar to $\triangle ABC$. Consequently, $\triangle AFE$ is a Kepler triangle.

In the same way, since DF is parallel to CA, it follows that $\triangle BDF$ is similar to $\triangle BCA$, and therefore that $\triangle BDF$ too is a Kepler triangle.

It remains to check $\triangle CED$ and $\triangle DEF$. We shall show that they too are Kepler triangles.

Consider $\triangle CED$ first. Noting that $\measuredangle C$ is shared by the two triangles, we must prove that $\triangle CED \sim \triangle CBA$. For this we must prove that

$$\frac{CE}{CB} = \frac{CD}{CA}, \quad \text{i.e.,} \quad CE \cdot CA = CD \cdot CB.$$

Now observe that

$$CE \cdot CA = z \cdot z(1+\varphi) = z^2 \cdot (1+\varphi),$$
$$CD \cdot CB = y\varphi \cdot y(1+\varphi) = y^2 \cdot \varphi \cdot (1+\varphi).$$

Equality follows since $z = y\sqrt{\varphi}$. It follows that $\triangle CED$ is a Kepler triangle.

To show that $\triangle DEF$ is a Kepler triangle, we only have to note that $DFEC$ is a parallelogram, hence $\triangle DEF$ is congruent to $\triangle EDC$; therefore $\triangle DEF$ too is a Kepler triangle.

It follows that by starting with a Kepler triangle and inscribing a triangle in it as described above, the inscribed triangles are also Kepler triangles. By repeating this process, we get an infinite array of nested Kepler triangles (see Figure 3). We may call them 'hidden Kepler triangles'.

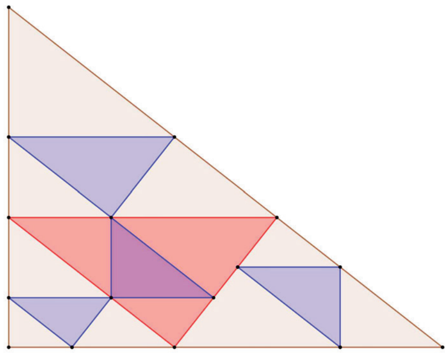

Figure 3.

Appendix: The golden section

Consider a line segment AB. We say that a point E on AB divides it in the **golden section** if $AE : EB = \varphi : 1$ or $1 : \varphi$.

Construction of point E. Suppose $AB = x$. Using a compass, construct AC perpendicular to AB, with $AC = x/2$. Join BC. With C as centre and radius AC, draw an arc intersecting BC at point D. With B as centre and radius BD, draw another arc intersecting AB at point E. Since $\triangle ABC$ is right-angled, $BC = \sqrt{5}x/2$, so $BD = BE = (\sqrt{5} - 1)x/2$. Hence $AE = (3 - \sqrt{5})x/2$. Taking ratios, we get $BE/AE = \varphi$, i.e., E divides AB in the ratio $1 : \varphi$ (Figure 4).

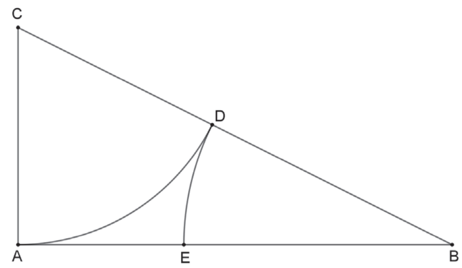

Figure 4.

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