Two Problems in Number Theory - Part II

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In this two-part article, we study two number theory problems from the UK Math Olympiad, Round 2, years 2006 and 2003 respectively. We had studied the first problem in the March 2021 issue, in Part I, and now we study the second problem in Part II. Both problems were discussed during meetings of the problem-solving group of our school.

Three-term arithmetic progressions with small prime divisors

Problem. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Find all triples (x, y, z) of distinct positive integers satisfying the following two conditions: (i) x, y, z are in an arithmetic progression; and (ii) $p(xyz) \leq 3$.

Solution. The condition $p(xyz) \leq 3$ implies that x, y, z are not divisible by any prime exceeding 3. This means that their only prime factors are 2 and 3.

Without loss of generality, we assume that the greatest common divisor of x, y, z is 1. For if $\gcd(x, y, z) = d$, where $d > 1$, then we may divide throughout by d and obtain a new set of triples satisfying the given conditions. On the other hand, starting with any triple x, y, z satisfying the conditions and with $\gcd(x, y, z) = 1$, we can obtain a new triple by multiplying each quantity by any fixed positive integer which does not have a prime factor greater than 3.

We may further assume that $x < y < z$.

Since $2y = x + z$, any common divisor of y with either x or z will be a divisor of the other. Hence $\gcd(x, y) = 1$ and $\gcd(y, z) = 1$. Combining this observation with the fact that x, y, z are divisible only by the primes 2 and 3, we are led to consider the following cases.

Keywords: Arithmetic progression, prime factor, greatest common divisor

Case 1: $x = 2^a, \quad y = 3^b, \quad z = 2^c$ for some non-negative integers a, b, c

Since x, y, z are in an A.P., $2^a + 2^c = 2 \cdot 3^b$, so

$$2^{a-1} + 2^{c-1} = 3^b.$$

If $a \geq 2$ and $c \geq 2$, then the left side is even whereas the right side is odd; so equality cannot hold. Therefore, one out of a and c must be 1. As we have assumed that $x < z$, it must be that $a = 1$. This gives $x = 2$. Substituting we get

$$1 + 2^{c-1} = 3^b.$$

Now we consider the following two sub-cases.

Sub-case 1a: $c - 1 = 1$

In this case we have $c = 2$ and $b = 1$, so $z = 4$ and $y = 3$. The triple in this case is $(2, 3, 4)$.

Sub-case 1b: $c - 1 \geq 2$

In this case we have

$$1 + 2^{c-1} = 3^b.$$

Since $c - 1 \geq 2$, it follows that 2^{c-1} is a multiple of 4. Also, $3 \equiv -1 \pmod 4$. Hence, reading both sides modulo 4, we get $1 \equiv (-1)^b \pmod 4$. Therefore b is even, i.e., $b = 2k$ for some integer $k \geq 0$.

Substituting in the original equation, we get $1 + 2^{c-1} = 3^{2k}$. Hence:

$$2^{c-1} = 3^{2k} - 1 = (3^k + 1) \cdot (3^k - 1).$$

Since the quantity on the left side is a power of 2, both factors on the right side (i.e., $3^k - 1$ and $3^k + 1$) must be powers of 2. As they are also consecutive even numbers, it follows that $3^k - 1 = 2$ and $3^k + 1 = 4$. (There is no other possibility.) Hence $k = 1$ and $b = 2$, giving $y = 3^b = 9$ and $z = 16$.

Thus in Case 1, we get the following two triples: $(2, 3, 4)$, $(2, 9, 16)$.

Case 2: $x = 3^a, \quad y = 2^b, \quad z = 3^c$ for some non-negative integers a, b, c

From $x + z = 2y$ we get

$$3^a + 3^c = 2^{b+1}.$$

If both $a, c > 0$, then the quantity on the left is a multiple of 3, but the quantity on the right cannot be a multiple of 3. Hence at least one out of a, c must be 0. Since $x < z$, we obtain $a = 0$. Hence $x = 1$. The equation now reads:

$$1 + 3^c = 2^{b+1}.$$

If $c = 0$, then we get $b = 0$ as well, and the triple in question is $(1, 1, 1)$. This obviously satisfies the given conditions.

If $c > 0$, then reading the equation modulo 3, we get:

$$1 \equiv (-1)^{b+1} \pmod 3.$$

This implies that $b+1$ is even, so $b = 2k-1$ for some positive integer k. From this we get:

$$1 + 3^c = 2^{2k},$$
$$\therefore\ 3^c = 2^{2k} - 1 = (2^k+1)\cdot(2^k-1),$$
$$\therefore\ 2^k + 1 = 3 \quad \text{and} \quad 2^k - 1 = 1.$$

This gives $k = 1$ and hence $b = 1$, $y = 2$, $z = 3$. The only solution in this case is $(1, 2, 3)$.

We had said at the start that we would be taking x, y, z to be coprime. Under this restriction, the only solutions to the problem are the triples $(1, 2, 3)$, $(2, 3, 4)$ and $(2, 9, 16)$. If we remove this restriction, we see that the solutions to the given problem are the triples $(d, 2d, 3d)$, $(2d, 3d, 4d)$ and $(2d, 9d, 16d)$, where d is any positive integer whose only prime factors are 2 and 3 (i.e., d is of the form $2^u \cdot 3^v$ for some non-negative integers u, v).

References

1. https://www.ukmt.org.uk

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