At Right Angles

A Resource for School Mathematics

ESTIMATION IN MATHEMATICS

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ESTIMATION IN MATHEMATICS

"It is the mark of an instructed mind to rest assured with that degree of precision that the nature of the subject admits, and not to seek exactness when only an approximation of the truth is possible." – Aristotle

Is mathematics always about exactness and accuracy? Does estimation have a role to play in mathematics?

Estimated answers are realistic enough to serve our purposes in most situations of daily life. There are clear benefits with regard to quickness in obtaining an adequate answer, for example in making a time plan for homework, in ordering food items at a restaurant, in planning a hike or a drive from one city to another, in the kitchen (judging the size of a jar needed for the cookies!).

There is no denial of the fact that estimation is a time-saving approach. It is useful in building physical intuition. It is also a valuable way of spotting errors and cross-checking answers arrived at by other means.

Does that mean that estimation is of use mainly in daily living? How do mathematicians and scientists use estimation in their work?

"No scientist ever treats a real-world problem exactly – just 'good enough' for the accuracy of the result they need." Astronomers attempting to determine movements of celestial objects cannot obtain precise measurements. Similarly for geologists attempting to find the mass and size of some underground object. Scientists have to make estimates of many measures using available data and modelling techniques.

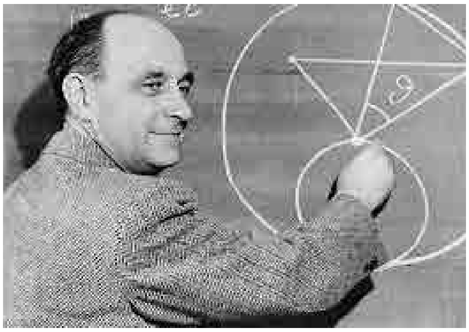

Figure 1

The physicist Enrico Fermi was famous for his ability to estimate various kinds of data with surprising precision.

WHAT IS ESTIMATION?

It is having a good guess at the size of something. But it is not a random guess. It makes use of prior experience and sound physical reasoning.

This reasoning involves usage of approximation, rounding, significant figures, scale factors, etc. And a good deal of common sense! It results in the computation of a value that is fairly close to the right answer.

Is there a right way to estimate? Does estimation need to be taught?

All of us have an innate capacity for estimation. If someone tosses a ball to us, we probably would know where to put our hands to catch the ball without thinking about it. If we had to select a line to stand in, we would estimate the number of people in each line and join the line with the least number of people. Or we may decide by the speed at which a line is moving. Yet, in many contexts of greater complexity, we need to practise the estimation skill to become good at it

Questions of estimation arise practically every day in our lives. Will this suitcase weigh more than 15 kg? How long will this download take? How many glasses can be filled by this jug of water?

Question: Seek more such examples from students from their daily life.

If you were a civil engineer you would make elaborate estimates of the quantity of the work and the expenditure on different works.

Estimations are context-based and the approaches used vary in different situations. However, there are some common techniques that are used in any estimation process. The estimation process involves an appreciation of the crucial variables and the way these variables interact.

Discussion: It is an interesting exercise to brainstorm with the students on examples of situations where exact answers are necessitated and where estimation would serve the purpose. A hospital nurse has to be precise in her measurement of a medicine.

Newspaper activity: Students can bring some newspaper cuttings of numerical information to class to discuss which ones involve exact numbers and which ones might be estimates.

Estimation is a highly neglected part of school mathematics. The ability to estimate can serve as a window into students' mathematical thinking and problem-solving. It is a high-level skill that requires students to be able to conceptualize and mentally manipulate numbers. Teachers should make use of estimation frequently in teaching and problem-solving sessions to help students build estimation skills. One needs to make a special effort to bring such problems into the classroom as there is not adequate focus given to this aspect in our textbooks.

Estimation skills have wide application in many subjects and it will be good to integrate mathematics with sciences and social sciences by choosing problems from all these subject areas.

Math talk: One can bring estimation questions into any topic in mathematics. It will be doubly good if we can ask the students to justify the reasons for their estimates. This will bring about math talk in the classroom and creates a space where peer learning can happen.

Figure 2

When teaching estimation teachers should use examples from life so that the children could try estimation based on the needs of the problem. They should also be ready to accept different solutions from students as long as they are reasonable. Also, students may occasionally know the answers to certain questions out of prior experiences. Teachers should be open and accept those answers. Allow students to share their personal experiences of estimation in daily life. Ultimately students need to know when to use which method, to get close to the precise value.

It is not necessary to give estimation problems as separate exercises. Often we require students to do some calculations. We can encourage students to make rough estimations before doing actual computations to check for the reasonableness of the answer.

Note: Teachers can point out to students that the guesswork that they do while dividing with double-digit or bigger numbers is estimation. Similarly, the square-root algorithm (which students may not be familiar with in upper primary school) also uses estimation.

Good estimation skills can be used as a tool in problem-solving and sensible approximations can simplify a problem to aid in focussing on the structure of the problem or patterns emerging from the given data of a problem.

Over or underestimation in crucial areas has a tremendous impact on any situation. India's overestimation of its production capacity for the Covid vaccine has created scarcity in the availability of vaccines. A lot of planning either at macro or micro levels is based on estimations of various things.

In this article, I have shared some crucial skills of estimation and varied estimation contexts which require their application. I have also shared questions framed by the famous Nobel Prize winner Fermi who posed open-ended estimation questions which challenge students' thinking.

SKILLS OF ESTIMATION

1. Estimation by comparison (with bodily measures, contrast with another object)
2. Using the prior sense of standard length, weight, volume which can serve as benchmarks for measurement
3. Scaling factor or ratios
4. Rounding
5. Usage of the highest place value
6. Choosing simpler fractions or decimals, estimating by halving or quartering
7. Modelling with simple shapes
8. Usage of averages
9. Adjustment of the answer based on rounding down or up
10. Estimation by sampling

ESTIMATION APPROACHES TO NON-ROUTINE QUESTIONS

The Nobel Prize winner Enrico Fermi posed questions that were open-ended, required numerical reasoning and generally came with limited information. His questions necessitated students to raise further questions and focus on the *process* of finding the solution rather than the answer itself.

They provoke curiosity and get students to think more creatively. The process of determining the answers requires students to brainstorm, discuss different approaches, debate them, find loopholes and refine them.

Here are some of his sample questions: "How many grains of sand are there on earth's beaches?" "How many atoms are there in the human body?"

How do we attempt questions of such a nature? We first write down any facts that we know related to the question. We look at possible procedures for determining the answer. We check the reasonableness of our answer

Example: How many hairs on a human head?

A question like this raises further questions. To start with what does a head resemble? A sphere. Assuming the head to be a sphere what is its radius? How much of it is covered with hair? What is the density of hair per square centimetre?

Here is another question. *"How many hours will you sleep in a lifetime?"*

The students begin to identify the variables in these problems as the first step.

How will they obtain the information that they need? Will they conduct an exhaustive survey? Will they test a sample group? Will they look for the relevant data on the internet?

They can then put down the formula or an algebraic expression that they have used while solving the problem.

Can they form some conclusions based on the results that they have got? Can they identify possible sources of error?

To attempt Fermi problems the class can be divided into groups so that each individual can participate actively in the discussion. The teacher plays the role of a guide, giving support as and when necessary.

1. ESTIMATION BY COMPARISON

1.1 Estimating by comparing with bodily measures

Children express quantities often in bodily terms though frequently exaggerated. In primary school, students can practise estimation of the lengths of various objects in the classroom by describing it as "The doormat is longer than my foot, it is twice my foot." Or: "The desk is shorter than me, it is half of my height."

Students of the upper primary school will be able to quantify the earlier statements with their prior awareness of the length of a ruler or their actual height. The descriptions would now be with estimated figures.

The ceiling must be 4 metres and the reasoning could be "If two of us students stand one over the other, there will be space for another half student. Each student is about one and a half metres."

Figure 3

1.2 Estimate by contrasting with another object

This is a common method that is frequently used by comparing the object to be measured with another object about which one has prior knowledge.

"The doormat must be about 60 cm long as two long rulers can fit on it."

In the same example of estimating the height of the ceiling, another student may estimate the ceiling to be four metres high. "The door is about two and a half metres and above the door, we can fit half of another door."

Activity with picture cards: Give students picture cards that show people and buildings, a group of people, people and tall structures like buildings, poles, etc.

Let students make intelligent guesses about the heights of various people in the picture and other structures. Students should be encouraged to justify their guesses.

Ensure that they keep perspective in mind and do not make the mistake of comparing nearby objects with objects that are at a distance, say a hill or trees in the background.

Question: Pose problems: "If this table is 2 feet long, how long is the bench?" "If this book is 4 cm thick, how thick will the dictionary be?"

If a table is 8 feet by 7 feet, what would it look like? Let students draw two or three different rectangles on the board and discuss the best solution.

In Figure 4 can you estimate the height of the tree?

Figure 4

Game: Expert estimate!!

Take a wound up nylon rope roll and ask the students to estimate the rope's total length. The student who makes the best estimate is the winner.

Figure 5

2. USING THE PRIOR SENSE OF STANDARD LENGTH, WEIGHT AND VOLUME WHICH CAN SERVE AS BENCHMARKS FOR MEASUREMENT

All of us have an intuitive sense of certain measures that are frequently used like 10 cm, 30 cm, 1 metre, 5 ml, 200 ml, 1 litre and 1 kg perhaps.

Give students pictures of various objects which are commonly used for estimation of capacity and weight.

Create a table to record the information. At the end students can measure actual capacity, lengths, and weights and compare their estimates with the actual figures.

Object	Estimate	Actual	Difference
Flag pole	6 metres	6.25 metres	.25 metre

Let's take the context of a picnic outing. We will need to carry food, fruits, water and some balls.

How many water bottles (standard bottles) can this crate hold?

Figure 6

Will 6 melons fit in the crate?

Figure 7

How many apples can it hold?

How many ice cubes can this bucket hold?

Figure 8

How many tennis balls can this bucket hold?

Pose questions: "If a banana weighs 45 grams what would be the weight of a guava fruit?"

How heavy would the rucksack be if I filled it with apples?

Figure 9

How many notebooks can my school bag hold?

If I fill it up completely, how heavy will my bag be?

Figure 10

3. SCALING FACTOR OR RATIOS

Solving an estimation problem requires enlargement and reduction.

It makes use of proportion and a scaling factor. Students will attempt to solve an equivalent problem with smaller values in order to obtain an answer to the original problem.

Example: How many books are there on these bookshelves?

Students would observe that the shelves hold roughly the same number of books. They would estimate or make an exhaustive count for one row of books or for the one-foot length of the shelf.

They would then multiply it by the number of such units to estimate the total number of books in the library.

This is an enlargement process.

Figure 11

Estimating bigger lengths using smaller lengths:

While measuring lengths or heights the unit varies from situation to situation. For instance, if we had to estimate the height of a multi-storeyed building, we would count the number of floors and multiply by the usual height of a single floor.

The height of a floor of the building can again be estimated from the height of a person standing near it or a doorway.

Here is a reduction process.

Estimating smaller length from a bigger length

At times we measure the total length of a stack or pile and divide by the appropriate number to arrive at the length of a unit.

Figure 12

Say, if we need to know the thickness of a wooden sheet we could measure the height of the stack and divide it by the number of sheets.

Example problems

1. A train terminal has 12 platforms. Each platform has about 115 trains departing each day. Find an estimate of the number of trains leaving the terminal every day.
2. If the current growth of population continues (get the latest data from the internet), what would be the population in India by 2030? Can we assume that this growth rate will stay the same? What can alter it?
3. At school, we are served milk and each one of us takes 2 full spoons of sugar with it. About how many kilograms of sugar do we use in the school every month?
4. Rahim can run 9.2 m per second. If he runs a 500m track what is your estimate of his timing?

Teachers can point out other mathematical concepts like factors and multiples which arise while doing these calculations.

4. ROUNDING

Rounding to the nearest tens, hundreds or powers of ten is employed to simplify the problems.

Students would probably have been taught by now the basics of rounding. In whole numbers, say 235, 236, 237, 238, 239 are rounded up to 240 while 231, 232, 233, 234 are rounded down to 230 (to the nearest tens). This can be shown on a number line for greater clarity.

Students should be taught to use rounding to estimate answers. They can also use rough answers to check calculations that they do either with paper-and-pencil or calculator. This would reveal any absurd errors that they might have made.

But students must also understand that rounding makes numbers easier to use – but at the cost of loss of precision.

If we are working with large numbers say in thousands it is easier to round the numbers to the nearest thousand for estimating answers. Choice of rounding is dependent on the level of precision needed. Close rounding makes calculations more precise. 2435 can be rounded to 2400 if we wish to retain hundreds or to 2000 if we don't.

In general, estimating involves rounding key quantities so that they can be manipulated easily. Rounding makes calculations easier. If we had to compute 31 x 49 it is easier to round it to 30 x 50 and get an approximate answer of 1500. The actual answer is 1519.

The same type of rounding can be applied to adding, subtracting, or dividing numbers as well.

- 83 × 31 = ?
- 39 + 97 = ?
- 83 – 57 = ?

Example: Sufi said that 523×34 gives 177,820. Is that a reasonable answer?

If students use the estimation of 500×30 to arrive at 15,000, they will quickly realize that the place value is way off.

If we have to work with decimal numbers we start by rounding the numbers to the nearest whole number. A number with a decimal value of 0.5 or higher is rounded up (for example, 1.8 becomes 2). A number with a decimal value lower than 0.5 is rounded down. (For example, 4.3 becomes 4).

Example: If a big parcel weighs 1.89 kg and a smaller parcel weighs 0.99 kg, what is the estimate for the total weight of 10 big parcels and 5 small parcels?

Rounding the big parcel to 2 kg and the small parcel to 1 kg would make the total 25 kg.

Intervals

Students can also use their understanding of the principles of rounding to determine the interval of the answer. That determines the lower and the upper-level figures for a given number.

Examples

1. "The number of pins in a box is rounded to the nearest fifty. If the number of pins is given as 2650 find the smallest and largest number of pins that could be in the box."

   The lower limit of this range would be 2625 and the higher limit will be 2674.
2. The number 11,200 has been rounded to the nearest one hundred. In what range does the actual number lie?
3. A number, when rounded to the nearest ten, becomes 400. In what range does the actual number lie?
4. Rohan makes a quick estimate of how many people weighing 72 kg can safely fit into a lift with a maximum load warning of 940 kg. He does this by rounding the values used in the calculation. What was his answer?

5. USAGE OF THE HIGHEST PLACE VALUE

When we sum amounts or evaluate products we use the highest place value as that gives us a rough idea of the total amount. Estimation with numbers reinforces the understanding of place value. Students realise that in two-digit numbers, the tens' digit has greater bearing than the ones' digit in contributing to the answer.

219 + 345 + 564 would be summed as 200 + 300 + 500 and hence the sum is above 1000.

What is the rough value of 2453 × 312?

It is taken as 2000 × 300, so the product is 600000. The number of zeroes have to be placed right.

How does one attempt a question like the one below?

Estimate the approximate answer of 53687 + 8365 + 1638 + 28.

Example: The average amount of storage space needed for a photograph taken with a phone is 3940 KB. The available storage space on a laptop is 217 GB. By rounding these numbers estimate the number of photographs that could be stored on a laptop.

The space for the photograph can be rounded to 4000 KB. The storage space can be taken as 200 GB that is 200,000,000 KB. Hence the number of photographs that could be stored is 50,000!

6. CHOOSING SIMPLER FRACTIONS OR DECIMALS, ESTIMATING BY HALVING OR QUARTERING

A good understanding of fraction concepts is helpful in selecting appropriate fractions and adjusting the result.

Example. If we had to calculate five-eighth of 4,100 we would choose 1/2 as it is slightly less than 5/8 and compute 1/2 of 4000 which is 2000. Adjusting for the reduced fractional part of 1/8 we can increase it by 1/4 of 2000, i.e., by 500. Our estimate is 2500. The actual answer is 2562.5.

Estimating by halving and quartering

Heights of buildings, trees and electric poles are often difficult to estimate and in such cases, we contrast them with nearby objects and use the process of halving and quartering to estimate the heights of tall structures as explained in the example problem here.

Figure 13

Example: In Figure 13 the vehicle which is at a distance seems to be slightly less than one half of the height of the lamp post.

The vehicle's height can be estimated roughly as 9 feet. The lamp post is probably around 20 feet.

In Figure 14 a man is standing on the scaffolding reaching the top of the pole. If we imagine quartering the pole the man's height corresponds to a quarter.

The pole height must be roughly 25 feet.

Figure 14

7. MODELLING WITH SIMPLE SHAPES

Complex or irregular shapes are approximated to a simple shape for estimation purposes.

How many round cakes fit into a cuboid box?

Figure 15

For estimation purposes, we treat the round cakes as squares and calculate.

How many melons fit into a carton? In order to estimate the number of melons that can fit into the carton, we model the melon with an appropriate sized cube or cylinder and evaluate how many such cubes or cylinders will fit into the carton.

Figure 16

Using simple shapes like cubes, cuboids helps in the estimation process.

8. USAGE OF AVERAGES

(**Note:** Average does not necessarily equal arithmetic mean)

Here is the data of money collected by ten students at the school fair.

55, 75, 60, 45, 85, 60, 65, 55, 55, 70

What is the rough sum of the amount collected?

We notice that the figures hover around an average figure of 60. Hence the sum is roughly 60 × 10=600.

9. ADJUSTMENT OF THE ANSWER BASED ON ROUNDING DOWN OR UP

While doing estimations we round figures up or down. That affects the answer and in order to address any major discrepancy that might arise we adjust the answer after the computation.

Example: 44 × 41 = ?

We may round it as 40 × 40 obtaining 1600. But 44 has been rounded down as 40 and hence we adjust the answer and make it 1800.

10. ESTIMATION BY SAMPLING

One technique that biologists use to estimate the number of species in a particular area is random sampling. It can be used to estimate population size. In this procedure, the organisms in a few small areas are counted and projected to the entire area. It is the same process that we use to answer the question "How many hairs on a human head?"

A biologist collected 50 litres of pond water and counted 10 mosquito larvae. How many larvae would you estimate to be in that pond if the total volume of water in the pond was 80,000 litres?

What are some difficulties with this technique? What could affect its accuracy?

11. TIME ESTIMATION

Time estimation skills are very crucial for students and everyone else.

Students can start with simple estimation like how long would it take to complete 10 math problems?

At the airport, you are 30th in a line. How long will you need to wait?

How many hours do you spend on video games in a month?

Very often our time estimates go wrong as we do not factor in two crucial elements.

- How long has it taken us to do a similar task in the past?
- Anticipating unexpected delays.

Help students to make realistic time estimates.

As a second step, the students can learn to estimate the time they spend on doing homework in various subjects and learn to plan and create a schedule for themselves.

Some estimation problems from various topics

Numbers

Example: How will you estimate the solution to this subtraction problem? 217 – 96 – 46

From 200 you can remove a 100 and a 50. That leaves 50 as your estimate.

How many times would 71 divide 423 roughly?

What is 23% of 123?

Between which two whole numbers is $\sqrt{2021}$?

What would you approximate $\sqrt{(1250/10000)}$ to?

If 7 cones cost Rs 280 what is the best estimate for the cost of 100 cones?

Which of the following is the best approximation for $\sqrt{108}$?

8, 9, 10, 11, 12

Fractions and decimals

- $\frac{91^2}{9.9} - \left(35 - \frac{7.4}{0.12}\right) = ?$
- $13 \times 0.2 = ?$
- $\frac{36}{0.6} = ?$
- $468 \times 7.9849 + 71 + 38 = ?$
- $42 \times \frac{21}{1.77} = ?$
- $304 \times \frac{0.736}{0.099} = ?$
- $0.31^2 = ?$

Angles

Estimate the measures of angles between your fingers when you spread them out. Is it the same for everyone?

Too often we see students read protractor in the wrong manner and come up with absurd answers. Estimation can largely avoid this problem.

Ability to estimate angles grows by using clocks and circular fraction kits.

Ideally this should have happened in the primary school.

Figure 17

Mensuration

Estimations with regard to capacity or total area are based on estimations of length, width, circumference and height.

Take two identical rectangular papers. Roll one into a short cylinder and the other into a tall cylinder. Estimate which one will hold more popcorn (if it had a base!).

Can you estimate the mass of a tree?

Can you estimate the number of mangoes on a tree?

Figure 18

There is a pile of waste material from the building site which needs to be disposed of. The pile is about 6 feet high in the middle and about 9 feet across.

Can a small truck carry this rubbish?

Figure 19

How many bricks were used for your school building?

Figure 20

Graphs

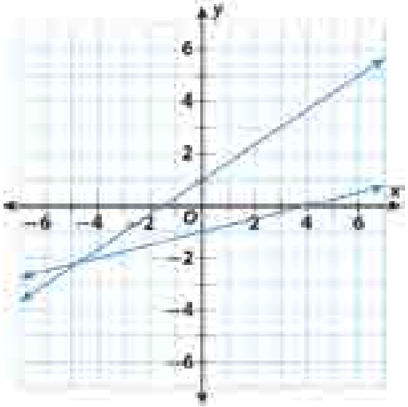

Figure 21

Example: Estimate the solution by sketching a graph of each linear function. Then solve the system algebraically. Use your estimate to judge the reasonableness of your solution.

$$x - 4y = 4$$

$$2x - 3y = -3$$

Example: Here is a pie chart of different types of books in a library. The blue part is fiction, the yellow is classics, the green portion is reference, and the red portion is encyclopaedias.

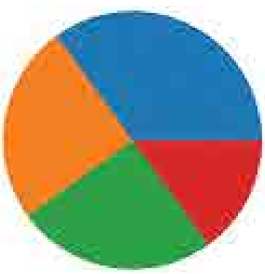

Figure 22

If the total number of books is 10,000 can you estimate the number of books of each type?

Example: How many soft drinks do you have in a month? Does it change over the year? Is it safe to consume such a quantity of soft drinks? Find out.

Example: How many sets of clothes do we buy in a month? What does your estimate show? Can you compute how much money we spend on clothes each year?

Example: Here is a child mortality graph of children in India over a period of time. What is your estimate for 2050?

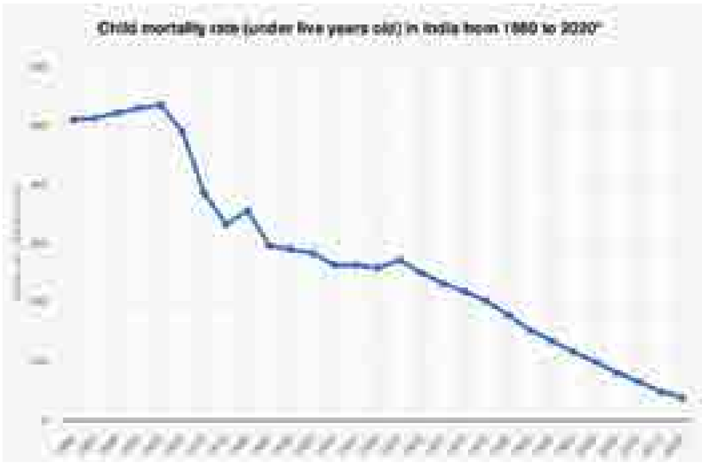

Figure 23

Fermi question bank

1. What distance will a ballpoint pen write?
2. How much paper is used at our school each week?
3. How much money is spent in the school canteen each day? In a week? Over the year?
4. How big a block of chocolate could you make using all the chocolate eaten by the class in a year?
5. What is the weight of garbage thrown away by each family every year?
6. How many children are needed to have a mass the same as an elephant?

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